NEW METHODS FOR MAGIC TOTAL LABELINGS OF GRAPHS

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Abstract

A *vertex magic total (VMT) labeling* of a graph $G = (V, E)$ is a bijection from the set of vertices and edges to the set of numbers defined by $\lambda : V \cup E \rightarrow \{1, 2, \ldots, |V| + |E|\}$ so that for every $x \in V$ and some integer k, $w(x) = \lambda(x) + \sum_{y:xy \in E} \lambda(xy) = k$. An *edge magic total (EMT) labeling* is a bijection from the set of vertices and edges to the set of numbers defined by $\lambda : V \cup E \rightarrow \{1, 2, \ldots, |V| + |E|\}$ so that for every $xy \in E$ and some integer k, $w(xy) = \lambda(x) + \lambda(y) + \lambda(xy) = k$. Numerous results on labelings of many families of graphs have been published. In this thesis, we include methods that expand known VMT/EMT labelings into VMT/EMT labelings of some new families of graphs, such as unions of cycles, unions of paths, cycles with chords, tadpole graphs, braid graphs, triangular belts, wheels, fans, friendships, and more.

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Chapter 1

Introduction

In order to understand graph labelings, it is important to know basic terminology and notations that will be used in this thesis. Most of these definitions can be found in [6], [24], [16], [11] or [21].

1.1 Basic Graph Terminology

Definition 1.1.1. A *graph* G consists of a finite set V of objects called *vertices* together with a set E of unordered pairs of vertices; the elements of E are called *edges*.

Definition 1.1.2. For every pair of vertices $x, y \in V$ that are connected by an edge $e \in E$, we call x and y the *endpoints* of e, and x and y are said to be *adjacent* to each other. The edge e also denoted by xy.

Definition 1.1.3. The set of all vertices adjacent to x is called the *neighborhood* of x, and denoted $N(x)$.

Definition 1.1.4. An edge that connects a vertex to itself is called a *loop*.

Definition 1.1.5. The *degree* of a vertex x in a graph, denoted $deg(x)$, is defined as the number of edges that are incident with x, with each loop counted twice.

Definition 1.1.6. A *simple graph* is a graph that does not contain any loops or multiple edges.

From here onward, all graphs mentioned in this thesis are simple graphs.

Definition 1.1.7. A graph G is called *r-regular* if for every $x \in V(G)$, $deg(x) = r$ for some positive integer r.

Definition 1.1.8. A *walk* in a graph G is a finite sequence of vertices $x_0, x_1, \ldots, x_n$ and edges $a_1, a_2, \ldots, a_n$ of G: $x_0, a_1, x_1, a_2, \ldots, a_n, x_n$, where the endpoints of a_i are x_{i-1} and x_i for each i.

Definition 1.1.9. A *path* P_n is a walk with n vertices in which no vertex is repeated. The *length* of a path is its number of edges.

Definition 1.1.10. The *distance* between two vertices in a graph is the number of edges in a shortest path connecting them.

Definition 1.1.11. A *cycle* C_n is a graph obtained by connecting the vertices v_1 and v_n in P_n using an edge e_n.

Definition 1.1.12. A *complete graph* K_n is a graph with n vertices in which every pair of distinct vertices is connected by an edge.

Definition 1.1.13. Given a graph $G = (V(G), E(G))$ with n vertices. The complement of G, denoted $\overline{G}$, is the graph with vertex set $V(\overline{G}) = V(G)$ and edge set $E(\overline{G}) = E(K_n) - E(G)$.

Definition 1.1.14. A *chord* is an edge joining two non-adjacent vertices in a cycle. The *length* of a chord is the distance between its endpoints on the cycle.

Definition 1.1.15. ${}^tC_n$ is a cycle C_n with one chord of length t.

In [15] the notation C_n^t is used for a cycle C_n with one chord of length t, but that notation is more commonly used for denoting a cycle C_n with chords connecting every pair of vertices of C_n that have distance t, which will not be discussed in this thesis.

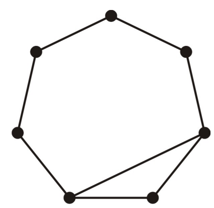


Figure 1.1: A cycle with one chord of length 2 $\left({}^{2}C_7\right)$

Definition 1.1.16. ${}^{[c]t}C_n$ is a cycle C_n with c chords, each of length t.

Definition 1.1.17. An (n, t)-*tadpole* (also known as *kite* or *dragon*) graph is the graph formed by joining the end point of a path P_t to a cycle C_n. The path and the cycle are called the *tail* and the *body* of the tadpole, respectively.

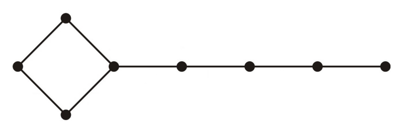


Figure 1.2: $(4, 4)$-tadpole graph

Definition 1.1.18. A *mutated* (m, n, t)-*tadpole* graph is the tadpole with m tails so that the vertices which are shared by the tails and the cycle are equidistant from each other.

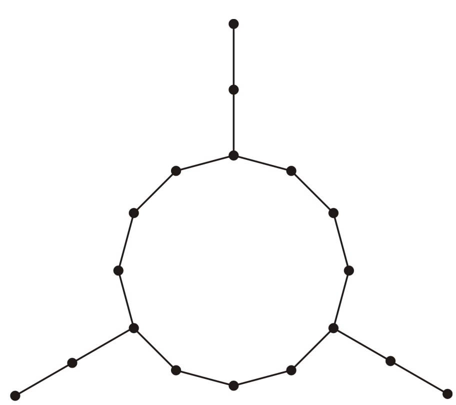


Figure 1.3: Mutated $(3, 12, 2)$-tadpole graph

Definition 1.1.19. For each $n \geq 3$, the *braid graph* $B(n)$ is a graph with vertex set $V = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ and edge set $E = \{(x_i x_{i+1}), (y_i y_{i+1}), (x_i y_{i+1}) \,|i = 1, 2, \ldots, n-1\} \cup \{(y_i x_{i+2}) \,|i = 1, 2, \ldots, n-2\}$.

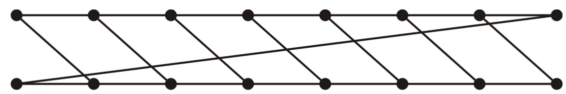


Figure 1.4: Braid graph $B(8)$

To define the next family of graphs, let $S = \{\uparrow, \downarrow\}$ be the set of symbols representing the position of the block as shown in Figure 1.5.

Figure 1.5: Block represented by arrow

Definition 1.1.20. Let α be a sequence of n symbols of S, i.e. $\alpha \in S^n$. Construct a graph by tiling n blocks side by side with their positions indicated by α. Denote the resulting graph by $TB(\alpha)$ and call it a *triangular belt*. For simplicity $(\uparrow, \uparrow, \ldots, \uparrow)$ is denoted by $(\uparrow^n)$ and $(\downarrow, \downarrow, \ldots, \downarrow)$ is denoted by $(\downarrow^n)$.

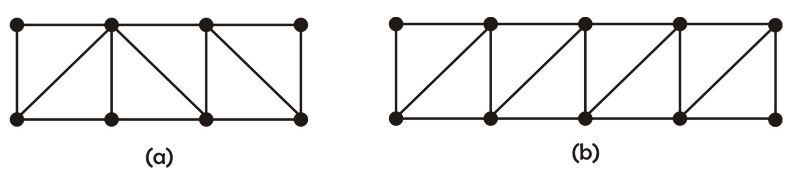


Figure 1.6: Triangular belts (a) $TB(\uparrow, \downarrow, \downarrow)$ and (b) $TB(\uparrow^4)$

In [24] the triangular belt $TB(\downarrow^n)$ is called *triangular ladder* $\mathbb{L}_n$.

Definition 1.1.21. A *wheel* W_n is the graph obtained by joining every vertex of the cycle C_n to exactly one isolated vertex called the center.

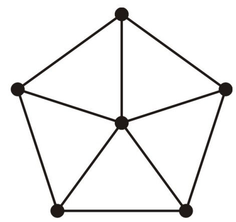


Figure 1.7: Wheel W_5

Definition 1.1.22. A *fan* $F_{m,n}$ is a graph obtained by connecting each vertex of path P_n to m isolated vertices.

In this thesis we will only consider the fan graph $F_{1,n}$. For simplicity we denote this graph by f_n, and call the only further vertex the *center* of the fan.

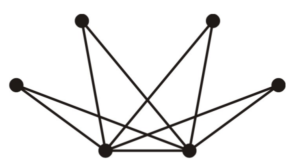


Figure 1.8: Fan $F_{4,2}$

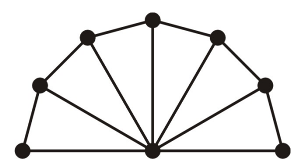


Figure 1.9: Fan f_6

Definition 1.1.23. For any integer $m \geq 3$ and $n \geq 2$, the *umbrella graph* $U(m, n)$ is a graph obtained by connecting the center of a fan f_n with a path of length $m - 1$ using an edge.

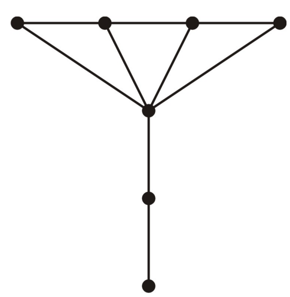


Figure 1.10: Umbrella $U(4, 3)$

Definition 1.1.24. A *friendship graph* Fr_n consists of n triangles C_3 with exactly one common vertex called the *center*.

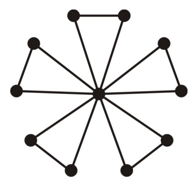


Figure 1.11: Friendship graph Fr_5

In [22], Slamin et al. denote friendship graphs with T_n. In [6], the friendship graph Fr_n is denoted by C_3^n, and called *Dutch n-windmill*. However, T is more commonly used for different family of graph known as *trees*, while the notation for a Dutch n-windmill is too similar with our notation for a cycle with chords. In order to avoid ambiguity, in this thesis we will use the notation Fr_n.

Definition 1.1.25. For integers $m, n \geq 0$ the *jellyfish graph* $J(m, n)$ is a graph with vertex set $V = \{u, v, x, y\} \cup \{x_1, x_2, \ldots, x_m\} \cup \{y_1, y_2, \ldots, y_n\}$ and edge set $E = \{(u, x), (u, v), (u, y), (v, x), (v, y$ $\{(x, x_i)|i = 1, \ldots, m\} \cup \{(y, y_i)|i = 1, \ldots, n\}$.

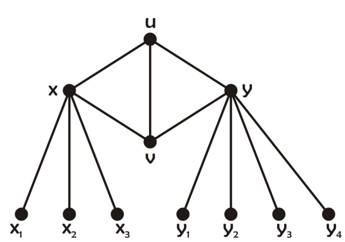


Figure 1.12: Jellyfish $J(3, 4)$

Definition 1.1.26. For an odd t, let $G \cong {}^{[t]2t}C_{4t}$ be a cycle with t chords (v_i, v_{i+2t}) for $i = 1, 3, 5, \ldots, 2t - 1$. *Rambutan* $R(t, m, n)$ is a graph obtained by connecting m pendant edges to each vertex which subscript equivalent to $2 \bmod 4$, and n pendant edges to each vertex which subscript equivalent to $0 \bmod 4$, of the graph ${}^{[t]2t}C_{4t}$.

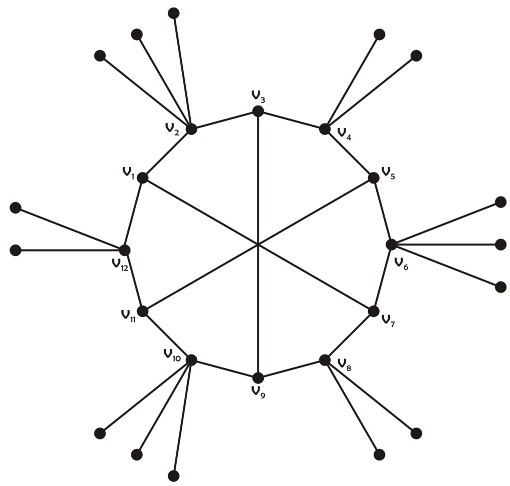


Figure 1.13: Rambutan $R(3,3,2)$

Definition 1.1.27. $P_n(+)N_m$ is a graph obtained by joining the endpoints of path P_n with m further vertices so that $V\left(P_n(+)N_m\right) = \{v_1, v_2, \ldots, v_n, y_1, y_2, \ldots, y_m\}$ and $E\left(P_n(+)N_m\right) = E(P_n) \cup \{(v_i, y_j)\}$, where $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$.

Figure 1.14 shows an example of the graph $P_4(+)N_3$.

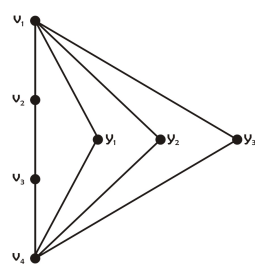


Figure 1.14: $P_4(+)N_3$

Definition 1.1.28. For odd values of n, the graph $C_{(2r+1)n}[+](2r+1)N_m$ is the graph obtained by joining some vertices of cycle C_n to $(2r+1)$ copies of m further vertices so that $V\left(C_{(2r+1)n}[+](2r+1)N_m\right) = \{v_1, v_2, \ldots, v_n\} \cup \{y_{11}, y_{12}, \ldots, y_{1m}\} \cup \{y_{21}, y_{22}, \ldots, y_{2m}\} \cup \cdots \cup \{y_{(2r+1)1}, y_{(2r+1)2}, \ldots, y_{(2r+1)m}\}$ and $E\left(C_{(2r+1)n}[+](2r+1)N_m\right) = E(C_n) \cup \{(v_i, y_{jk}), (y_{jk}, v_{i+2})\}$ where $i \equiv 1 \pmod{(2r+1)}$, $j = \frac{2r+i}{2r+1}$ and $k = 1, 2, \ldots, m$.

Figure 1.15 shows an example of the graph $C_9[+]2N_3$.

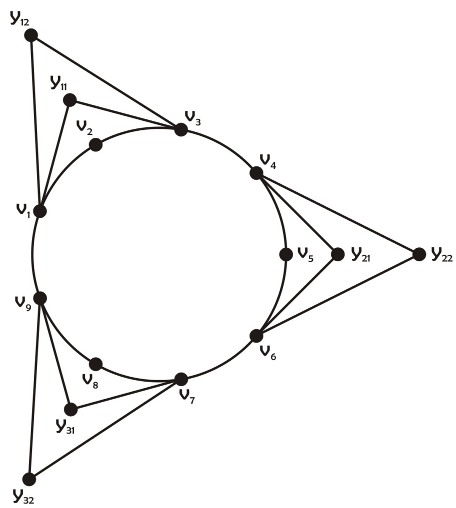


Figure 1.15: $C_9[+]3N_2$

1.2 Unions, Products and Joins of Graphs

Definition 1.2.1. The *union* $G \cup H$ of graphs G and H is the graph consisting the sets $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. The union of m identical graphs G is denoted by mG.

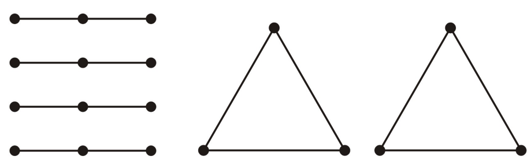


Figure 1.16: $4P_3 \cup 2C_3$

Definition 1.2.2. The *Cartesian product* $G \,\square\, H$ of graphs G and H is a graph such that the vertex set of $G \,\square\, H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices

(u, u') and (v, v') are adjacent in $G \square H$ if and only if either $u = v$ and u' is adjacent to v' in H, or $u' = v'$ and u is adjacent to v in G.

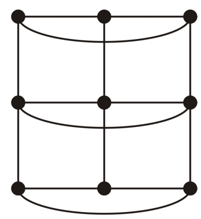


Figure 1.17: $P_3 \square C_3$

The notation $G \times H$ is occasionally also used for Cartesian products of graphs, but is more commonly used for another construction known as the *tensor product* or *direct product* of graphs, which will not be discussed in this thesis. The square symbol is the more common and unambiguous notation for the Cartesian product of graphs.

Definition 1.2.3. A *book graph* B_n is the Cartesian product $S_{n+1} \square P_2$.

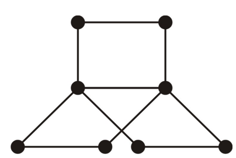


Figure 1.18: B_3

Definition 1.2.4. For any integers $m \geq 3$ and $h \geq 2$, the *Mongolian Ger* $M(m, h)$ is the graph obtained by connecting each vertex in the first row of $P_h \square C_m$ to an isolated vertex called the *peak* of the ger.

For odd values of m, let x denote the peak of $M(m, h)$ and $v_1, v_2, \ldots, v_m$ denote the vertices in the first row of $P_h \square C_m$. The *incomplete Mongolian Ger* $MT(m, h)$ is a graph obtained by removing edges $xv_2, xv_4, \ldots, xv_{m-1}$.

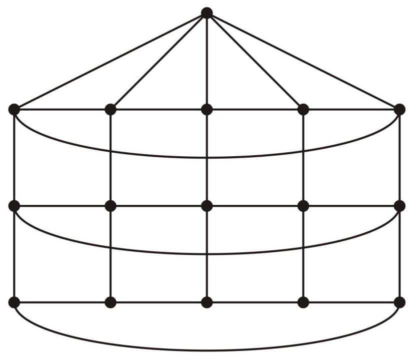


Figure 1.19: $M(5,3)$

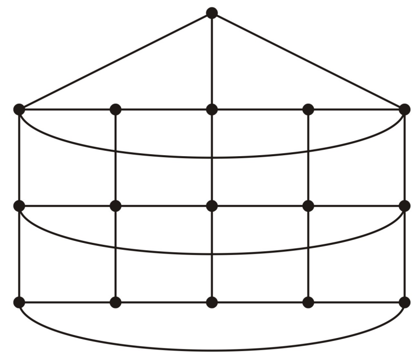


Figure 1.20: $MT(5,3)$

Definition 1.2.5. The *lexicographic product* or *graph composition* $G \circ H$ or $G[H]$ of graphs G and H is a graph such that the vertex set of $G \circ H$ is the Cartesian product $V(G) \times V(H)$, and any two vertices (u, v) and (x, y) are adjacent in $G \circ H$ if and only if either u is adjacent to x in G or $u = x$ and v is adjacent to y in H.

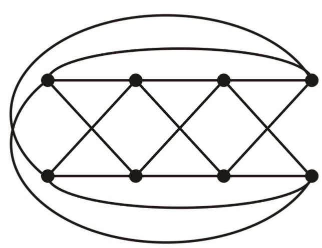


Figure 1.21: $C_4 \circ \overline{K}_2$

In this thesis we will use the notation $G \circ H$.

Definition 1.2.6. The *crown product* $G \odot H$ of graphs G and H is the graph that is obtained by placing a copy of one G and $|V(G)|$ copies of H so that all vertices in the same copy of H are joined with exactly one vertex of G while each vertex of G is joined to all vertices of exactly one copy of H.

Definition 1.2.7. A *join* of graphs G and H, denoted by $G + H$, is the graph with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$.

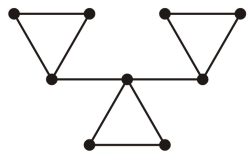


Figure 1.22: $P_3 \odot P_2$

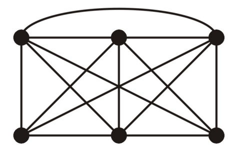


Figure 1.23: $P_3 + C_3$

1.3 Graph Labeling

Definition 1.3.1. A graph *labeling* is a mapping from a set of elements of a graph (can be vertices, edges, or a combination of both) to a set of numbers (usually positive integers).

There are various types of graph labelings. To avoid confusion, in this section we mention some commonly known types of labeling. Most information in this section is taken from [6]. The labeling we are working on, called the magic labeling, will be explained later in Section 1.4.

1.3.1 Rosa's Valuation and Graceful Labeling

Rosa [19] called a function f a β*-valuation* of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. Golomb [8] called such labelings *graceful* and the term has become the standard one.

A graph G with q edges is said to be *graceful* if there is an injection f from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that the labels of the edges are every possible difference of the vertex labels, that is, the set $\{1, 2, \ldots, q\}$.[1]

There are many variations of graceful labelings, which are described in [6]. We will not discuss this matter any further as they are not our main interest in this thesis.

1.4 Magic Labeling

All information in this section is taken from [24].

Magic labelings were introduced by Sedláček [20] in 1963. There are many variations of magic labeling. In this thesis we focus only on several types of magic labeling.

Definition 1.4.1. A *total labeling* of a graph is a graph labeling with the set of vertices and edges as the domain.

Definition 1.4.2. Under a total labeling λ, the *weight* of a vertex $x \in V$ is defined as $w(x) = \lambda(x) + \sum_{y \in N(x)} \lambda(xy)$, while the weight of an edge $xy \in E$ is defined as $w(xy) = \lambda(x) + \lambda(y) + \lambda(xy)$.

Definition 1.4.3. A *vertex magic total (VMT) labeling* of a graph $G = (V, E)$ is a bijection from the set of vertices and edges to the set of numbers defined by $\lambda : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ so that for every $x \in V$ and some integer k, $w(x) = k$. The constant k is called the *magic constant* for λ.

Definition 1.4.4. A *super vertex magic total (SVMT) labeling* of a graph $G = (V, E)$ is an VMT labeling λ that has the property $\lambda(V) = \{1, 2, \ldots, |V|\}$ and $\lambda(E) = \{|V| + 1, |V| + 2, \ldots, |V| + |E|\}$.

Definition 1.4.5. An *edge magic total (EMT) labeling* of a graph $G = (V, E)$ is a bijection from the set of vertices and edges to the set of numbers defined by $\lambda : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ so that for every $xy \in E$ and some integer k, $w(xy) = k$.

Definition 1.4.6. A *super edge magic total (SEMT) labeling* of a graph $G = (V, E)$ is an EMT labeling λ that has the property $\lambda(V) = \{1, 2, \ldots, |V|\}$ and $\lambda(E) = \{|V| + 1, |V| + 2, \ldots, |V| + |E|\}$.

Definition 1.4.7. A graph G that is admits both VMT and EMT labelings is called *totally magic*.

1.5 Permutation Cycles and Kotzig Arrays

Definition 1.5.1. A *permutation* of a set A is a bijection from A to A.

It is customary to take A to be a set of the form $\{1, 2, 3, \ldots, n\}$ for some positive integer n. For example we define a permutation set $\{1, 2, 3, 4\}$ by specifying

$$\tau(1) = 3, \quad \tau(2) = 1, \quad \tau(3) = 4, \quad \tau(4) = 2$$

This function can be expressed in an array form as

$$\tau = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{bmatrix}$$

One notation that is commonly used to specify permutations is called *cycle notation* and was first introduced by Cauchy in 1815. This cycle notation writes the mapping in continuous circular motion 1 to 3, 3 to 4, 4 to 2 and 2 back to 1. The example above can be written as the cycle permutation $\tau = (1, 3, 4, 2)$.

Definition 1.5.2. [7] An *s-cycle* is a permutation cycle of the form $(a_1, a_2, \ldots, a_s)$.

Let $\tau_1 = (1, 3, 4, 2)$ and $\tau_2 = (1, 4, 2, 3)$. The composition $\tau_2\tau_1$ first takes 1 to 3 (in τ_1), then 3 back to 1 (in τ_2). It takes 2 to 1, then 1 to 4. The number 4 is mapped to 2 then 2 to 3, and 3 is mapped to 4, then 4 to 2. Thus $\tau_2\tau_1 = (1)(2, 4, 3)$. The composition $\tau_1\tau_1$ first takes 1 to 3, then 3 to 4. It takes 4 to 2, then 2 to 1. The number 3 is mapped to 4 then 4 to 2, and 2 is mapped to 1, then 1 to 4. Thus $\tau_1\tau_1 = \tau_1^2 = (1, 4)(2, 3)$.

If s is odd, then a composition of two identical s-cycles will preserve the length s. For example let $\tau = (1, 3, 5, 4, 2)$, then we have $\tau^2 = (1, 5, 2, 3, 4)$. In general, if τ is an s-cycle and $\gcd(a, s) = 1$, then τ^a also an s-cycle. Hence when s is odd, both τ and τ^2 are s-cycles.

Definition 1.5.3. A *Kotzig array* is a $d \times m$ grid, each row being a permutation of $\{0, 1, \ldots, m - 1\}$ and each column having the same sum.

The Kotzig array used in this thesis is the $3 \times (2r+1)$ Kotzig array that is given as an example in [16]:

$$\begin{bmatrix} 0 & 1 & \dots & r & r+1 & r+2 & \dots & 2r \\ 2r & 2r-2 & \dots & 0 & 2r-1 & 2r-3 & \dots & 1 \\ r & r+1 & \dots & 2r & 0 & 1 & \dots & r-1 \end{bmatrix}$$

Definition 1.5.4. A *modified Kotzig array* is a $d \times m$ grid, which is obtained by adding 1 to every element and switching the second and third rows of a Kotzig array.

1.6 Other Definitions

In addition to the definitions in the previous sections, in this section we give definitions of some other notation and terminology. We do not have any result about them in this thesis, but the known results about them related to vertex and edge magic labeling are included in the Appendix.

Definition 1.6.1. A *tree* is a connected graph that does not contain a cycle.

Definition 1.6.2. A tree is called a *rooted tree* if one vertex has been designated the root, in which case the edges have a natural orientation, towards or away from the root. In a rooted tree, the *parent* of a vertex is the vertex adjacent to it on the path to the root; every vertex except the root has a unique parent. A *child* of a vertex v is a vertex of which v is the parent.

Definition 1.6.3. A *n-ary tree* is a rooted tree in which each vertex has at most n children. 2-ary trees are also called *binary* trees.

Definition 1.6.4. A *caterpillar* is a tree with the property that the removal of all vertices of degree one leaves a path.

Definition 1.6.5. A *star graph* S_n is a tree on n vertices with one vertex having degree $n-1$ and the other $n-1$ vertices having degree 1. The star graph S_n is isomorphic to the complete bipartite graph graph $K_{1,n-1}$.

In his survey [6] Gallian uses notation $St(a_1, a_2, ..., a_n)$ to denote the disjoint union of n stars $S_{a_1}, S_{a_2}, \ldots, S_{a_n}$.

Definition 1.6.6. A *bipartite graph* is a graph whose vertices can be divided into two disjoint sets V_1 and V_2 such that every edge connects a vertex in V_1 to vertex in V_2. A *complete* bipartite graph is a bipartite graph where every vertex of V_1 is connected to every vertex of V_2.

The notation of a bipartite graph with $|V_1| = m$ and $|V_2| = n$ is $K_{m,n}$.

Definition 1.6.7. A *unicyclic graph* is a graph containing exactly one cycle.

Definition 1.6.8. A *helm* H_n is the graph obtained from a wheel by attaching a pendant edge at each vertex of the n-cycle.

Definition 1.6.9. The *Petersen graph* is the graph with 10 vertices and 15 edges, shown in Figure 1.24.

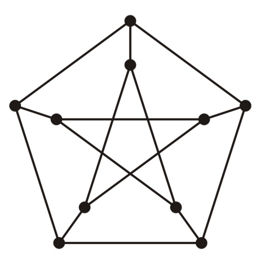


Figure 1.24: Petersen graph

Definition 1.6.10. An (n, e)-*graph* is a graph G with $|V(G)| = n$ and $|E(G)| = e$.

Chapter 2

Related Results

In 1963 magic labelings were first studied. Naturally, a number of variations were created. There are numerous types of magic labelings in graph theory. In this thesis we just focus on VMT, EMT, SVMT and SEMT labelings. Hundreds of papers have been published regarding these labelings.

In this chapter we list the known results about these four labelings that are related to the new results we found. Sections 2.1 to 2.4 include the results on cycles and unions of cycles, which are 2-regular graphs. The other sections give other variatons of cycles, which are not regular graphs.

For 2-regular graphs, we can obtain an EMT (SEMT) labeling from a VMT (SVMT) labeling. Supose λ is a VMT (SMVT) labeling of a 2-regular graph G with n vertices, then the EMT (SEMT) labeling λ' of G can be obtained by shifting each label one unit clockwise so that for $1 \leq i \leq n$, $v_i \in V(G)$ and $e_i \in E(G)$, we have $\lambda(v_i) = \lambda'(e_i)$ and $\lambda(e_i) = \lambda'(v_{i+1})$, where the subscript is taken modulo n.

Figure 2.1 describes how to obtain an EMT labeling from a VMT labeling of C_4.

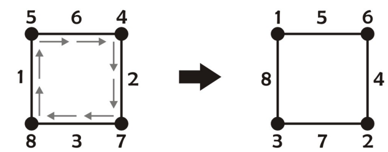


Figure 2.1: VMT and EMT labeling of C_4 with $k = 12$

2.1 Cycles

Theorem 2.1.1. [13] *C_n has a VMT labeling for any $n \geq 3$.*

Theorem 2.1.2. [14] *C_n has an SVMT labeling if and only if n is odd.*

Theorem 2.1.3. [16] *If n is odd, then C_n has an SEMT labeling with $k = \frac{1}{2}(5n + 3)$ and with $k = \frac{1}{2}(7n + 3)$.*

Theorem 2.1.4. [16] *If n is even, then C_n has an EMT labeling with $k = \frac{1}{2}(5n + 4)$ and with $k = \frac{1}{2}(7n + 2)$.*

2.2 Unions of Identical Cycles mC_n

Theorem 2.2.1. [23] *mC_n has an SVMT labeling if and only if m and n is odd.*

Theorem 2.2.2. [4] *The 2-regular graph mC_n has an SEMT labeling if and only if m and n are odd.*

Theorem 2.2.3. [16] *The 2-regular graph mC_n has a VMT labeling if m is odd.*

Corollary 2.2.4. *The 2-regular graph mC_n has an SEMT labeling if and only if m is odd.*

More results on other variations of cycles are provided in a tabular form in the Appendix. The known results for mC_n are summarized in the following table:

m	n	Magic Total Labeling
odd	odd	SVMT, SEMT
odd	even	VMT, EMT, not SVMT, not SEMT
even	odd	not EMT for $n = 3$, for $n > 3$ are unkown
even	even	unknown

One of our results in Section 3.3 concerns the case when m and n both even.

2.3 Unions of Two Cycles of Distinct Length $C_{n_1} \cup C_{n_2}$

All results in this section are taken from [3]. For convenience and consistency in this thesis, we changed the notation from the original source.

Theorem 2.3.1. *The 2-regular graph $G \cong C_{n_1} \cup C_3$ has an SEMT labeling if and only if $n_1 \geq 6$ and n_1 is even.*

Theorem 2.3.2. *The 2-regular graph $G \cong C_4 \cup C_{n_2}$ has an SEMT labeling if and only if $n_2 \geq 5$ and n_2 is odd.*

Theorem 2.3.3. *The 2-regular graph $G \cong C_{n_1} \cup C_5$ has an SEMT labeling if and only if $n_1 \geq 4$ and n_1 is even.*

Theorem 2.3.4. *If n_1 is even with $n_1 \geq 4$ and n_2 is odd with $n_2 \geq \frac{n_1}{2} + 2$, then the 2-regular graph $G \cong C_{n_1} \cup C_{n_2}$ has an SEMT labeling.*

Corollary 2.3.5. *For $n_1 = 6, 8$ or 10, the 2-regular graph $C_{n_1} \cup C_{n_2}$ has an SEMT labeling if and only if $n_2 \geq 3$ and n_2 is odd.*

The preceding results lead the authors in [3] to pose the following conjecture.

Conjecture 2.3.1. The 2-regular graph $C_{n_1} \cup C_{n_2}$ has an SEMT labeling if and only if $n_1 + n_2 \geq 9$ and $n_1 + n_2$ is odd.

Similar results are given in Section 3.4 for vertex magic labelings.

2.4 Unions of Cycles $C_n \cup hC_3$

The results in this section are taken from [10]. For convenience and consistency in this thesis, we again changed the notation from the original source.

Theorem 2.4.1. *For each positive integer $t \geq 3$, the disjoint union $C_4 \cup (2t-1)C_3$ has an SVMT labeling, with magic constant $k = 21t + 5$.*

Theorem 2.4.2. *For each positive integer $t \geq 2$, the disjoint union $C_5 \cup (2t)C_3$ has an SVMT labeling, with magic constant $k = 21t + 19$.*

Theorem 2.4.3. *For each positive integer t, the disjoint union $C_7 \cup (2t)C_3$ has an SVMT labeling, with magic constant $k = 21t + 26$.*

The preceding results lead the authors in [10] to make the following conjecture.

Conjecture 2.4.1. A 2-regular graph of an odd order possesses an SVMT labeling if and only if it is not one of $C_4 \cup C_3, C_4 \cup 3C_3$ or $C_5 \cup 2C_3$.

An expansion of these results, as well as partial contribution to Conjecture 2.4.1 are given in Section 3.5.

2.5 Unions of Cycles and Paths $C_{n_1} \cup P_{n_2}$

All results in this section are taken from [3].

Theorem 2.5.1. *For every integer n_2 with $n_2 \geq 6$, the 2-regular graph $H \cong C_3 \cup P_{n_2}$ has an SEMT labeling.*

Theorem 2.5.2. *The graph $H \cong C_4 \cup P_{n_2}$ has an SEMT labeling if and only if $n_2 \neq 3$.*

Theorem 2.5.3. *For every integer $n_2 \geq 4$, the 2-regular graph $H \cong C_5 \cup P_{n_2}$ has an SEMT labeling.*

Theorem 2.5.4. *If n_1 is even with $n_1 \geq 4$ and $n_2 \geq \frac{n_1}{2}+2$, then the graph $H \cong C_{n_1} \cup P_{n_2}$ has an SEMT labeling.*

Corollary 2.5.5. *For every positive integer n_2, the graph $C_{n_1} \cup P_{n_2}$ has an SEMT labeling when $n_1 = 4, 5, 6, 8$ or 10, unless $(n_1, n_2) = (4, 3), (6, 1), (10, 1)$.*

An expansion of these results is given in Section 4.2.

2.6 A Cycle with a Chord ${}^tC_n$

All results in this section are taken from [15].

Theorem 2.6.1. *A cycle with one chord ${}^tC_{4m+1}$ has an SEMT labeling for all t other than $t = 5, 9, 4m-4, 4m-8$ given $m \geq 3$.*

Theorem 2.6.2. *A cycle with one chord ${}^tC_{4m+1}$ has an SEMT labeling for all $t \equiv 1 \pmod 4$ except $t = 4m-3$.*

Theorem 2.6.3. *A cycle with one chord ${}^tC_{4m}$ has an SEMT labeling for $t \equiv 2 \pmod 4$.*

Theorem 2.6.4. *A cycle with one chord ${}^tC_{4m+2}$, $m \geq 2$ has an SEMT labeling for $t = 2, 6$ and all odd t other than 5.*

An expansion of these results is given in Section 4.3.

2.7 Lexicographic Product $C_n \circ \overline{K}_2$

In [12] Kotzig and Rosa stated the next result for an EMT labeling of a complete bipartite graph. They used the terminology M-valuation, which is now known as EMT labeling.

Theorem 2.7.1. [12] *An M-valuation [EMT labeling] of the complete bipartite graph $K_{p,q}$ exists for all $p, q \geq 1$.*

Observe that $K_{4,4} \cong C_4 \circ \overline{K}_2$, so an EMT labeling for $C_4 \circ \overline{K}_2$ exists.

In Section 4.4 we give an expansion of this result to get an EMT labeling of $C_n \circ \overline{K}_2$ for greater values of n and also the labeling for the unions of these lexicographic products.

2.8 Cartesian Product $P_m \,\square\, C_n$

Results in this section are taken from [26] (published in Bahasa Indonesia, the national language of Indonesia) and from [2]. In some publications, the graph $P_2 \,\square\, C_n$ might be referred to as a *prism* graph.

Theorem 2.8.1. [26] *If n is odd then the graph $P_m \,\square\, C_n$ has an EMT labeling with magic constant $k = (3m - \frac{1}{2})n + \frac{3}{2}$ and with magic constant $k = (6m - \frac{5}{2})n + \frac{3}{2}$.*

Theorem 2.8.2. [2] *The generalized prism $P_m \,\square\, C_n$ has an SEMT labeling if n is odd and $m \geq 2$.*

Theorem 2.8.3. [26] *The graph $P_2 \,\square\, C_n$ does not have an EMT labeling for $n \equiv 2 \pmod 4$.*

One of our results in Section 4.5 will give an alternative way to find the EMT labeling of $P_m \,\square\, C_n$ for other value of n from the EMT labeling with smaller values of n (not using the definition given in [26] or [2]).

2.9 Cartesian Product $P_2 \,\square\, P_n$

In some publications, the graph $P_2 \,\square\, P_n$ is referred to as the *ladder* graph.

Theorem 2.9.1. [26] *If n is odd then the graph $P_2 \,\square\, P_n$ has an EMT labeling with magic constant $k = \frac{1}{2}(11n + 1)$ and with magic constant $k = \frac{1}{2}(19n - 7)$.*

Theorem 2.9.2. [26] and [2]. *If n is odd then the ladder $L_n \cong P_2 \,\square\, P_n$ has an SEMT labeling with magic constant $k = \frac{1}{2}\,(11n + 1)$.*

The converse of Theorem 2.9.2 is not true since the ladder $L_n \cong P_2 \,\square\, P_n$ can have an SEMT labeling when n is even.

Results about this graph will be discussed in Section 4.6.

2.10 Unions of Paths

Results in this section are taken from [3], [4] and [5].

Theorem 2.10.1. [3] *The graph* $F \cong P_m \cup P_n$ *has an SEMT labeling if and only if* $(m, n) \neq (2, 2)$ *or* $(3, 3)$*.*

Theorem 2.10.2. [4] *Union of paths* mP_n *has an SEMT labeling if* m *is odd.*

Theorem 2.10.3. [4] *Union of paths* $P_3 \cup mP_2$ *has an SEMT labeling for all* m*.*

Theorem 2.10.4. [4] *Union of paths* $m(P_2 \cup P_n)$ *has an SEMT labeling if* m *is odd and* $n \in \{3, 4\}$*.*

In [5], the special case of Theorem 2.10.1 when $m = n$ is written as in Theorem 2.10.5 below.

Theorem 2.10.5. [5] $2P_n$ *has an SEMT labeling if and only if* n *is not* 2 *or* 3*.*

Theorem 2.10.6. [5] $2P_{4n}$ *has an SEMT labeling for all* n*.*

An expansion of these results is given in Section 4.7.

2.11 (n, t)-tadpole graph

Results in this section are taken from [16], [25] and [18].

Theorem 2.11.1. [16] *An* $(n, 1)$*-tadpole has an EMT labeling.*

Theorem 2.11.2. [25] *An* $(n, 1)$*-tadpole has an SEMT labeling when* n *is odd.*

Theorem 2.11.3. [18] *An* $(n, 2)$*-tadpole has an SEMT labeling if and only if* n *is even.*

Some EMT and SEMT labelings of tadpoles for different values of n and t, as well as how to obtain EMT/SEMT labeling for mutated tadpoles from these results are given in Section 4.8

2.12 Friendship graph Fr_n, Fan f_n, Wheel W_n

All results in this section and are taken from [22].

Theorem 2.12.1. *The friendship graph* Fr_n *has an SEMT labeling if and only of* $n \in \{3, 4, 5, 7\}$.

Theorem 2.12.2. *For* $n \geq 2$*, every fan* f_n *has an EMT labeling with magic constant* $k = 4n + 2$.

Theorem 2.12.3. *For* $n \equiv 6 \pmod 8$*, every wheel* W_n *has an EMT labeling with magic constant* $k = 5n + 2$.

Expansions of the results for friendship, fan and wheel graphs are given in Sections 4.9, 4.10 and 4.11, respectively.

The rest of the results in this chapter, that is, for Section 2.13, 2.14 and 2.15, are all taken from [21].

2.13 $P_{2n}(+)N_m$ and $(P_2 \cup mK_1) + N_2$

Theorem 2.13.1. *The graph* $P_{2n}(+)N_m$ *has an SEMT labeling for all* $n, m \geq 1$.

Theorem 2.13.2. *For* $m \geq 1$*, the planar graph* $(P_2 \cup mK_1) + N_2$ *has an SEMT labeling.*

We show how to use Theorem 2.13.1 to obtain an SEMT labeling for the graph $C_{(2n+1)(2r+1)}[+]N_m$ is shown in Section 4.13. An expansion of Theorem 2.13.2 is given in Section 4.14.

2.14 Braid $B(n)$, Triangular Belt $TB(\alpha)$, Umbrella $U(m,n)$

Theorem 2.14.1. *The braid graph $B(n)$ has an SEMT labeling for all $n \geq 3$.*

Theorem 2.14.2. *For any $\alpha \in S^n, S = \{\uparrow, \downarrow\}, n > 1$, the triangular belt $TB(\alpha)$ has an SEMT labeling.*

Theorem 2.14.3. *For any integer $m \geq 2$ and $n \geq 0$, the umbrella graph $U(m,n)$ has an SEMT labeling.*

Expansions of the results for braid, triangular belt and umbrella graphs are given in Sections 4.15, 4.16 and 4.12, respectively.

2.15 Jellyfish $J(m,n)$, Incomplete Mongolian Ger $MT(m,h)$

Theorem 2.15.1. *The jellyfish graph $J(m,n)$ has an SEMT labeling for all $m, n \geq 0$.*

Theorem 2.15.2. *For each odd $m \geq 3$ and $h \geq 2$, the incomplete Mongolian ger $MT(m,h)$ has an SEMT labeling.*

We show how to use Theorem 2.15.1 to obtain an SEMT labeling for the rambutan graph is given in Section 4.17. An expansion of Theorem 2.15.2 is given in Section 4.18.

Chapter 3

Results on Vertex and Super Vertex Magic Total Labelings

In this chapter we will introduce two methods that can be applied to 2-regular graphs. The first method preserves the VMT (SVMT) properties as we extend the length of a 2-regular graph, while the second method preserves VMT (SVMT) properties as we multiply the number (make copies) of a 2-regular graph, both by a factor of an odd number.

3.1 Method 1: Extending the Length of Cycles

The only members of 2-regular graphs family are cycles and unions of cycles. In this section we will show how the method is used to preserve the properties of vertex magic total labelings as we extend the length of each cycle of mC_n to $mC_{n(2r+1)}$, where r is a positive integer.

Let κ be the modified Kotzig array as defined in the introduction chapter.

$$\kappa = \begin{bmatrix} 1 & 2 & \dots & r+1 & r+2 & \dots & 2r & 2r+1 \\ r+1 & r+2 & \dots & 2r+1 & 1 & \dots & r-1 & r \\ 2r+1 & 2r & \dots & 1 & 2r-1 & \dots & 4 & 2 \end{bmatrix}$$

If we write the first two rows of κ as a permutation cycle τ, we have:

$$\tau = (1, r+1, 2r+1, r, 2r, \dots, 3, r+3, 2, r+2)$$

The difference between two consecutive elements in τ is equal to r taken modulo $(2r+1)$. Note that τ is a $(2r+1)$-cycle. Since $(2r+1)$ is an odd number for every nonnegative integer r, then $\gcd(2, 2r+1) = 1$, and so we have τ^2 also a $(2r+1)$-cycle. This fact plays an important role in preserving the properties of magic labeling of our VMT and SVMT labeling as we extend the length of the 2-regular graph.

Let κ' be the modified κ, where we switched the first and second row of κ:

$$\kappa' = \begin{bmatrix} r+1 & r+2 & \dots & 2r+1 & 1 & \dots & r-1 & r \\ 1 & 2 & \dots & r+1 & r+2 & \dots & 2r & 2r+1 \\ 2r+1 & 2r & \dots & 1 & 2r-1 & \dots & 4 & 2 \end{bmatrix}$$

Obviously, if we write the first two rows of κ' as a permutation cycle, we have τ^{-1}.

Let λ' be a VMT labeling for any 2-regular graph G. For every vertex and edge of G, let λ be the labeling obtained by decreasing the original label by 1, that is, let $\lambda(v) = \lambda'(v) - 1$ and $\lambda(e) = \lambda'(e) - 1$.

For each cycle in G, construct an $n \times 3$ table with entries as follows.

- In the first column: For $i = 1, 2, ..., n$, the entry in the i^{th} row is the 3×1 matrix
$$\Lambda = \begin{bmatrix} \lambda(e_i) \\ \lambda(e_{i+1}) \\ \lambda(v_{i+1}) \end{bmatrix}.$$

- In the second column: For $y = 1, 2, 3$ and $z = 1, 2, ..., (2r+1)$ the entry in the i^{th} row is either κ or κ' that is, $= \begin{cases} \kappa = [\kappa_{yz}], & \text{if } i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ \kappa' = [\kappa'_{yz}], & \text{if } \left\lfloor \frac{n}{2} \right\rfloor + 1 < i \leq n \end{cases}$

- In the third column: For $i = 1, 2, ..., n$, the entry in the i^{th} row is the matrix
$$\Theta_i = \begin{cases} [\kappa_{yz} + (2r+1)\Lambda_{y1}], & \text{if } i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ [\kappa'_{yz} + (2r+1)\Lambda_{y1}], & \text{if } \left\lfloor \frac{n}{2} \right\rfloor + 1 < i \leq n \end{cases}$$

If we multiply the permutation cycles of κ and κ' in the second column, we obtain $\tau^{\left\lfloor \frac{n}{2} \right\rfloor + 1} \tau^{n - \left(\left\lfloor \frac{n}{2} \right\rfloor + 2\right)} = \tau^{2\left\lfloor \frac{n}{2} \right\rfloor - n + 2}$. If n is odd we have $\tau^{(n-1)-n+2} = \tau$ and if n is even we have $\tau^{n-n+2} = \tau^2$.

The cycle $C_{n(2r+1)}$ is obtained by tracking the numbers on Θ. Let θ^i_{yz} denote the element of Θ_i in the y^{th} row and z^{th} column. In each Θ_i, the two numbers θ^i_{1z} and θ^i_{2z} will be the labels of two incident edges on $C_{n(2r+1)}$, and θ^i_{3z} will be the label of the vertex they share.

Observe that by how Θ is defined, we have $\left\{\theta^{i+1}_{1z}\right\} = \left\{\theta^i_{2z}\right\}$, that is, for all numbers on $\left\{\theta^{i+1}_{1z}\right\}$, there exists exactly one number with equal value on $\left\{\theta^i_{2z}\right\}$. They denotes the same edge on $C_{n(2r+1)}$, track the numbers consecutively taking i modulo n until all numbers in Θ_i are used. This is possible to do because both the possible values for the power of τ in the second column are relatively prime to $2r+1$ for any integer r.

For all $1 \leq i \leq n$, all pairs θ^i_{1z} and θ^i_{2z} will represent labels of incident edges. Since $1 \leq z \leq 2r+1$, the result is the extended cycle $C_{n(2r+1)}$.

Observe that in this process, by multiplying all λ by $(2r+1)$, the magic constant will be also multipled by $(2r+1)$, and by reducing all labels by one from λ' to λ, the magic constant will be reduced by $3r$. Hence after performing Method 1, the magic constant k will change to $(2r+1)k - 3r$.

3.2 Method 2: Multiplying the Number of Cycles

This method is similar to the theorem about disconnected graphs given in [16]. In this section we give a detailed description of this method for obtaining a VMT labeling of mC_n from a VMT labeling of $2C_n$, which will be used in the next sections.

A construction of a VMT labeling for mC_n with $m \equiv 2 \pmod 4$ can be obtained using the following steps.

- Make $\frac{m}{2}$ copies of labeled $2C_n$. From now we treat these copies as pairs.
- Make a $3 \times \frac{m}{2}$ Kotzig array, multiply each entry of the array by $4n$.
- Add the number in the i^{th} column of the first row to the labels of all edges with odd subscripts in the i^{th} pair of $2C_n$.
- Add the number in the i^{th} column of the second row to the labels of all edges with even subscripts in the i^{th} pair of $2C_n$.
- Add the number in the i^{th} column of the third row to the labels of the vertices in the i^{th} pair.

A construction of a VMT labeling for mC_n with $m \equiv 4 \pmod 8$ can be obtained using a similar method.

- Make $\frac{m}{4}$ copies of labeled $4C_4$. From now we treat these copies as a set of four.
- Make a $3 \times \frac{m}{4}$ Kotzig array, multiply each entry of the array by $12n$.
- Add the number in the i^{th} column of the first row to the labels of all edges with odd subscripts in the i^{th} set of $4C_4$.
- Add the number in the i^{th} column of the second row to the labels of all edges with even subscripts in the i^{th} set of $4C_4$.

- Add the number in the i^{th} column of the third row to the labels of the vertices in the i^{th} set.

3.3 Unions of an Even Number of Identical Even Cycles

Figures 3.1 – 3.4 gives the VMT labelings for $2C_n$ when $n \in \{4, 6, 8, 10\}$.

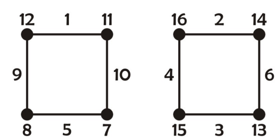


Figure 3.1: VMT labeling for $2C_4$

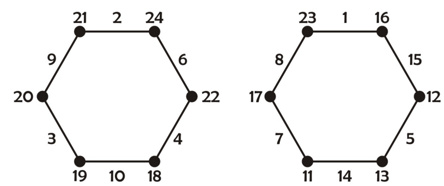


Figure 3.2: VMT labeling for $2C_6$

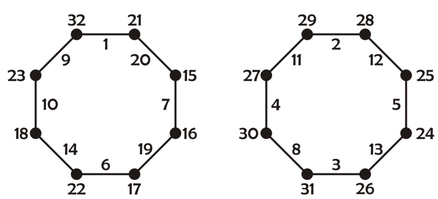


Figure 3.3: VMT labeling for $2C_8$

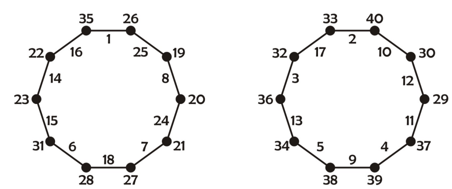


Figure 3.4: VMT labeling for $2C_{10}$

Based on these small cases, we make with a conjecture.

Conjecture 3.3.1. For even values of n, $2C_n$ has a VMT labeling with $k = 5n + 2$.

So far we have been unable to generalize the labeling. However, we can use Method 1 to extend the length of each cycle in $2C_n$ to construct a VMT labeling for $2C_{n(2r+1)}$, where r is a nonnegative integer. This is possible because the method for extending the length works separately for each single cycle. To see better how the methods works, we present two examples as follows.

Example: $2C_6 \to 2C_{18}$

In this example we perform Method 1 to extend $2C_6$ into $2C_{18}$, that is, when $2r+1=3$. From the labeling for $2C_6$ in Figure 3.2, construct a table for each cycle as shown here.

Λ	κ_i or κ_i'			Θ_i		
1	1	2	3	4	5	6
5	2	3	1	17	18	16
23	3	1	2	72	70	71
5	1	2	3	16	17	18
3	2	3	1	11	12	10
21	3	1	2	66	64	65
3	1	2	3	10	11	12
9	2	3	1	29	30	28
17	3	1	2	54	52	53
9	1	2	3	28	29	30
2	2	3	1	8	9	7
18	3	1	2	57	55	56
2	2	3	1	8	9	7
8	1	2	3	25	26	27
19	3	1	2	60	58	59
8	2	3	1	26	27	25
1	1	2	3	4	5	6
20	3	1	2	63	61	62

Λ	κ_i or κ_i'			Θ_i		
0	1	2	3	1	2	3
14	2	3	1	44	45	43
15	3	1	2	48	46	47
14	1	2	3	43	44	45
4	2	3	1	14	15	13
11	3	1	2	36	34	35
4	1	2	3	13	14	15
13	2	3	1	41	42	40
12	3	1	2	39	37	38
13	1	2	3	40	41	42
6	2	3	1	20	21	19
10	3	1	2	33	31	32
6	2	3	1	20	21	19
7	1	2	3	22	23	24
16	3	1	2	51	49	50
7	2	3	1	23	24	22
0	1	2	3	1	2	3
22	3	1	2	69	67	68

Table 3.1: Tables for $2C_6 \to 2C_{18}$

From each table we will construct one C_{18}. In the first table, the first column of Θ_1 implies 4 and 17 are the labels of incident edges and the label of the vertex they share is 72. The second column of Θ_2 implies 17 and 12 are the labels of incident edges and the label of the vertex they share is 64. Continuing in a similar way, from $\Theta_i, 1 \leq i \leq 6$ we get a path of length 6, with $(4, 17, 12, 28, 8, 25, 6)$ as the labels of the edges, in consecutive order.

Going back from Θ_6 to the third column of Θ_1, we have 6 and 16 are the labels of incident edges and the label of the vertex they share is 71, and so on. Continuing until all elements in all Θ_i are used, we get a cycle of length 18 in which the edges labeled in consecutive order with $(4, 17, 12, 28, 8, 25, 6, 16, 11, 30, 7, 27, 5, 18, 10, 29, 9, 26)$. Repeat similar steps for the second table. The VMT labeling for $2C_{18}$ with $k = 93$.

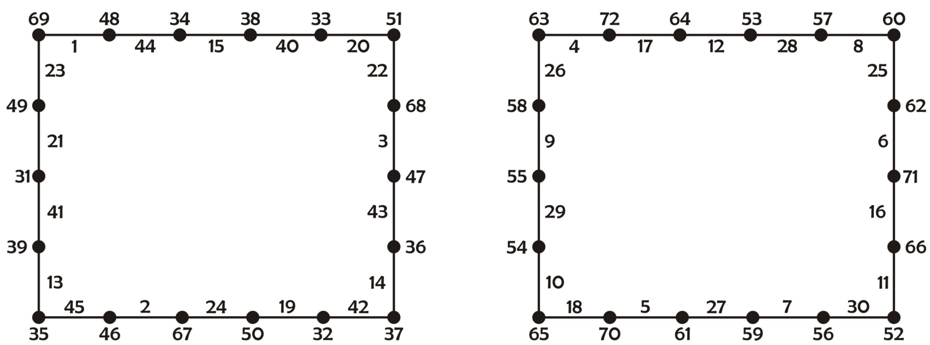


Figure 3.5: VMT labeling for $2C_{18}$

Example: $2C_4 \to 10C_4$

This example shows how to obtain a VMT labeling for $10C_4$ from the VMT labeling of $2C_4$ shown on Figure 3.1 using Method 2.

First make $\frac{10}{2} = 5$ copies of labeled $2C_4$ and a 3×5 Kotzig array, then multiply each entry of the array by $4 \cdot 4 = 16$.

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0

x 16

0	16	32	48	64
48	64	0	16	32
48	16	64	32	0

Figure 3.6: The Kotzig array for λ for $10C_4$

Next add the entries in the array to the label of edges and vertices of 5 copies of $2C_4$ as described in Method 1.

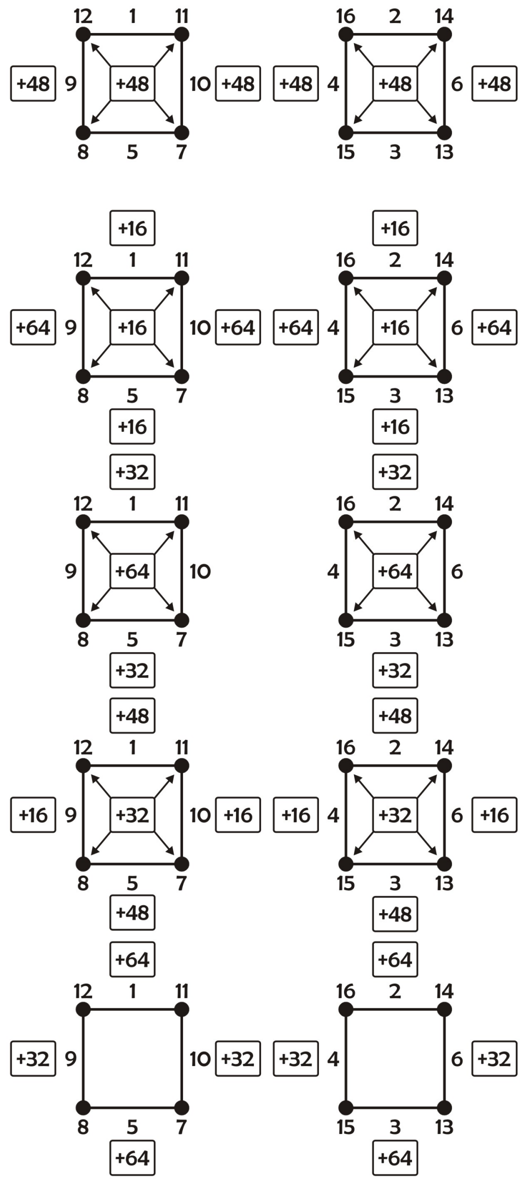


Figure 3.7: Adding entries in the Kotzig array to the original labels

Hence we have a VMT labeling for $10C_4$ as follows.

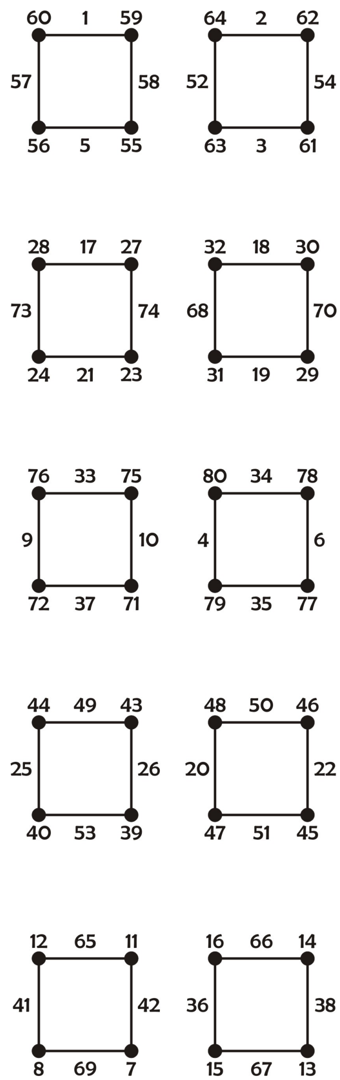


Figure 3.8: VMT labeling of $10C_4$ with $k = 118$

Theorem 3.3.1. *When* $m \equiv 2 \pmod 4$ *and either* $n \equiv 2 \pmod 4$, $n \equiv 4 \pmod 8$ *or* $n \equiv 8 \pmod{16}$, *the unions of cycles* $mC_{n(2r+1)}$ *have a VMT labeling with* $k = (5n+2)(2r+1) - 3r + (3mn - 6n)$.

Proof. The result follows from applying both Method 1 and Method 2 to the VMT labelings of $2C_n$ given in Figures 3.1, 3.2 and 3.3. ∎

In order to modify these methods for unions of even number of even cycles when $m \not\equiv 2 \pmod 4$, we will need to start from a VMT where $m = 2^a, a \geq 2$. For $a \geq 3$ the result is still unknown, but for $m \equiv 2 \pmod 4$ we can start using the VMT labeling for $4C_4$ below:

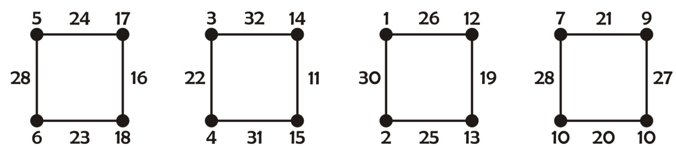


Figure 3.9: VMT labeling for $4C_4$

Theorem 3.3.2. *For $m \equiv 4 \pmod 8$ and $n \equiv 4 \pmod 8$, mC_n has a VMT labeling with magic constant $k = 57(2r+1) - 3r + 72\left(\frac{m}{4} - 1\right)$.*

Proof. The result follows from applying both Method 1 and Method 2 to the labeling in Figure 3.9. ■

3.4 Unions of Non-Identical Cycles $m\left(C_{n_1(2r+1)} \cup C_{n_2(2r+1)}\right)$

Method 2 can be modified for obtaining an SVMT labeling for $m\left(C_{n_1} \cup C_{n_2}\right)$ from $C_{n_1} \cup C_{n_2}$ from the known results in Section 2.3.

- Make m copies of labeled $C_{n_1} \cup C_{n_2}$, treat C_{n_1} and C_{n_2} as a pair in each copy.
- Make a $3 \times m$ Kotzig array, multiply each entry of the array by $2(n_1 + n_2)$.
- Add the number in the i^{th} column of the first row to the labels of all edges with odd subscripts in the i^{th} pair.
- Add the number in the i^{th} column of the second row to the labels of all edges with even subscripts in the i^{th} pair.

- Add the number in the i^{th} column of the third row to the labels of the vertices in the i^{th} pair.

For convenience call this Method 3. By combining this method with Method 1, we have a several new results.

Theorem 3.4.1. *For any nonnegative integer r and odd m, the 2-regular graph $m\left(C_{n_1(2r+1)} \cup C_{3(2r+1)}\right)$ has an SVMT labeling when $n_1 \geq 6$ and n_1 is even.*

Proof. Applying Method 1 to Theorem 2.3.1, we get an SVMT labeling of $C_{n_1(2r+1)} \cup C_{3(2r+1)}$ for any $r \geq 0$. Applying Method 3 to this labeling we get an SVMT labeling of $m\left(C_{n_1(2r+1)} \cup C_{3(2r+1)}\right)$ for odd value of m. ∎

Theorem 3.4.2. *For any nonnegative integer r and odd m, the 2-regular graph $m\left(C_{4(2r+1)} \cup C_{n_2(2r+1)}\right)$ has an SVMT labeling when $n_2 \geq 5$ and n_2 is odd.*

Proof. Applying Method 1 to Theorem 2.3.2, we get an SVMT labeling of $C_{4(2r+1)} \cup C_{n_2(2r+1)}$ for any $r \geq 0$. Applying Method 3 to this labeling we get an SVMT labeling of $m\left(C_{4(2r+1)} \cup C_{n_2(2r+1)}\right)$ for odd value of m. ∎

Theorem 3.4.3. *For any nonnegative integer r and an odd m, the 2-regular graph $m\left(C_{n_1(2r+1)} \cup C_{5(2r+1)}\right)$ has an SVMT labeling when $n_1 \geq 4$ and n_1 is even.*

Proof. Applying Method 1 to Theorem 2.3.3, we get an SVMT labeling of $C_{n_1(2r+1)} \cup C_{5(2r+1)}$ for any $r \geq 0$. Applying Method 3 to this labeling we get an SVMT labeling of $m\left(C_{n_1(2r+1)} \cup C_{5(2r+1)}\right)$ for odd value of m. ∎

Theorem 3.4.4. *If m is odd, n_1 is even with $n_1 \geq 4$ and n_2 is odd with $n_2 \geq \frac{n_1}{2} + 2$, then the 2-regular graph $m\left(C_{n_1(2r+1)} \cup C_{n_2(2r+1)}\right)$ has an SVMT labeling.*

Proof. Applying Method 1 to Theorem 2.3.4, we get an SVMT labeling of $C_{n_1(2r+1)} \cup C_{n_2(2r+1)}$ for any $r \geq 0$. Applying Method 3 to this labeling we get an SVMT labeling of $m\left(C_{n_1(2r+1)} \cup C_{n_2(2r+1)}\right)$ for odd value of m. ■

Corollary 3.4.5. *For $n_1 = 6, 8$ or 10, an odd m and any nonnegative integer r, the 2-regular graph $m\left(C_{n_1(2r+1)} \cup C_{n_2(2r+1)}\right)$ has an SVMT labeling if $n_2 \geq 3$ and n_2 is odd.*

From Conjecture 2.3.1, we also get the following conjecture.

Conjecture 3.4.1. For odd values of m, the 2-regular graph $m\left(C_{n_1} \cup C_{n_2}\right)$ is super vertex-magic if and only if $n_1 + n_2 \geq 9$ and $n_1 + n_2$ is odd.

Example: $C_3 \cup C_6 \to C_9 \cup C_{18}$

Shown below is the process of applying Method 1 to the known SVMT labeling for $C_3 \cup C_6$ given in [3] in order to obtain an SVMT labeling for $C_9 \cup C_{18}$. Note that $9 < \frac{18}{2} + 2$ so the SVMT labeling for $C_9 \cup C_{18}$ is indeed not included among the results in [3].

Start from an SVMT labeling for $C_3 \cup C_6$ given in [3]:

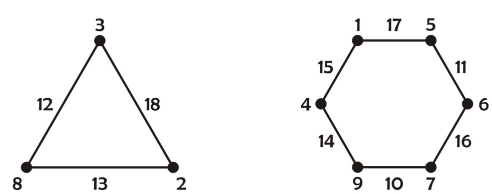


Figure 3.10: SVMT labeling for $C_3 \cup C_6$

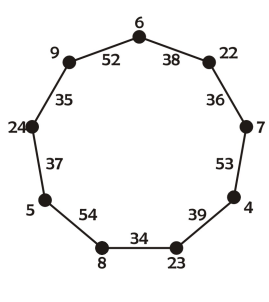
Making tables as described in Method 1 for C_3 and C_6, we have:

Λ	κ_i or κ_i'			Θ_i		
17	1	2	3	52	53	54
12	2	3	1	38	39	37
1	3	1	2	6	4	5
12	1	2	3	37	38	39
11	2	3	1	35	36	34
7	3	1	2	24	22	23
11	2	3	1	35	36	34
17	1	2	3	52	53	54
2	3	1	2	9	7	8

Λ	κ_i or κ_i'			Θ_i		
16	1	2	3	49	50	51
10	2	3	1	32	33	31
4	3	1	2	15	13	14
10	1	2	3	31	32	33
15	2	3	1	47	48	46
5	3	1	2	18	16	17
15	1	2	3	46	47	48
9	2	3	1	29	30	28
6	3	1	2	21	19	20
9	1	2	3	28	29	30
13	2	3	1	41	42	40
8	3	1	2	27	25	26
13	2	3	1	41	42	40
14	1	2	3	43	44	45
3	3	1	2	12	10	11
14	2	3	1	44	45	43
16	1	2	3	49	50	51
0	3	1	2	3	1	2

Table 3.2: Tables for $C_3 \cup C_6 \to C_9 \cup C_{18}$

From the table, we get the SVMT labeling for $C_9 \cup C_{18}$ as shown in Figure 3.11.



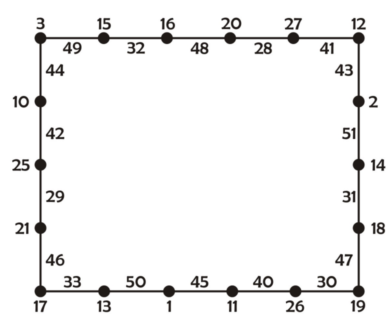


Figure 3.11: SVMT labeling for $C_9 \cup C_{18}$

All the results stated above focus on SVMT labelings. No results have been published for VMT labeling of $C_{n_1} \cup C_{n_2}$ when their lengths are both odd or both even. In this thesis we include a small result for VMT labeling for cases where $n_1 = 6$ and $n_2 = 4$. The graph $C_6 \cup C_4$ has a VMT labeling with $k = 27$.

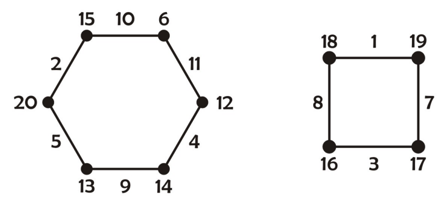


Figure 3.12: VMT labeling for $C_6 \cup C_4$

Applying Method 2 with $m = 3$, we get a VMT labeling for $3\,(C_6 \cup C_4)$ as shown on Figure 3.13.

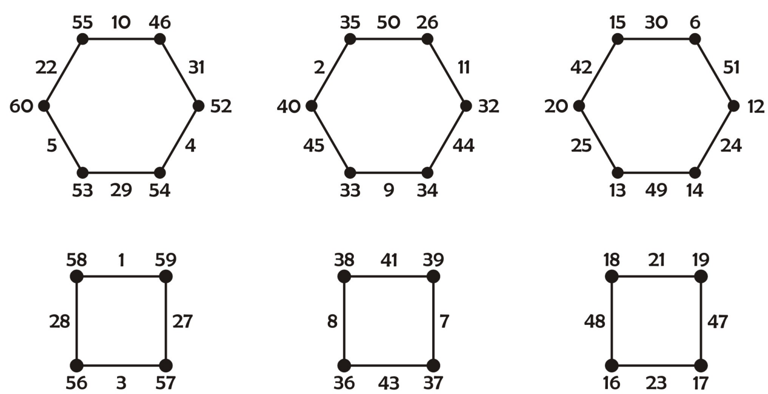


Figure 3.13: VMT labeling for $3\,(C_6 \cup C_4)$

Combining both Method 1 and Method 3 for $C_6 \cup C_4$, we have the theorem.

Theorem 3.4.6. *For any nonnegative integer r and an odd value of m, the union of cycles $m\left(C_{6(2r+1)} \cup C_{4(2r+1)}\right)$ has a VMT labeling.*

Proof. Applying Method 1 to the labeling shown in Figure 3.13, we get an SVMT labeling of $C_{n_1(2r+1)} \cup C_{5(2r+1)}$ for any $r \geq 0$. Applying Method 3 to this labeling we get an SVMT labeling of $m\left(C_{n_1(2r+1)} \cup C_{5(2r+1)}\right)$ for odd value of m. ■

3.5 Unions of Cycles $m\left(C_{n(2r+1)} \cup hC_3\right)$

Applying both Method 1 and Method 3 to the results listed in Section 2.4, we have several results that partially answer Conjecture 2.4.1.

Theorem 3.5.1. *For any nonnegative integer r, odd m and every positive integer $t \geq 2$, the disjoint union $m\left(C_{5(2r+1)} \cup 2tC_{3(2r+1)}\right)$ has an SVMT labeling.*

Proof. Applying Method 1 to Theorem 2.4.2, we get an SVMT labeling of $C_{5(2r+1)} \cup 2tC_{3(2r+1)}$ for any $r \geq 0$ and $t \geq 2$. Applying Method 3 to this labeling we get an SVMT labeling of $m\left(C_{5(2r+1)} \cup 2tC_{3(2r+1)}\right)$ for an odd value of m. ■

Theorem 3.5.2. *For any nonnegative integer r, odd m and every positive integer $t \geq 3$, the disjoint union $m\left(C_{4(2r+1)} \cup (2t-1)C_{3(2r+1)}\right)$ has an SVMT labeling.*

Proof. Applying Method 1 to Theorem 2.4.1, we get an SVMT labeling of $C_{4(2r+1)} \cup (2t-1)C_{3(2r+1)}$ for any $r \geq 0$ and $t \geq 3$. Applying Method 3 to this labeling we get an SVMT labeling of $m\left(C_{4(2r+1)} \cup (2t-1)C_{3(2r+1)}\right)$ for an odd value of m. ■

Theorem 3.5.3. *For any nonnegative integer r, odd m and every positive integer t, the disjoint union $m\left(C_{7(2r+1)} \cup 2tC_{3(2r+1)}\right)$ has an SVMT labeling.*

Proof. Applying Method 1 to Theorem 2.4.3, we get an SVMT labeling of $C_{7(2r+1)} \cup 2tC_{3(2r+1)}$ for any $r \geq 0$ and $t \geq 1$. Applying Method 3 to this labeling we get an SVMT labeling of $m\left(C_{7(2r+1)} \cup 2tC_{3(2r+1)}\right)$ for an odd value of m. ■

3.6 Further Research on VMT/SVMT Labelings

In [17], McQuillan gives a technique for constructing magic labelings of 2-regular graphs. In this section we will compare his results with the results that can be obtained using our methods. Theorem 3.6.1 is the result from [17], which gives a significant contribution to Conjecture 2.4.1 mentioned in Section 3.4.

Theorem 3.6.1. [17] *Let G be the disjoint union $G \cong C_{h_1} \cup C_{h_2} \cup \dots C_{h_l}$. Let $I = \{1, 2, \dots, l\}$ and J be any subset of I. Let $G_J = \left(\bigcup_{i \in J} nC_{h_i} \right) \cup \left(\bigcup_{i \in I-J} C_{nh_i} \right)$ where all unions are disjoint unions and n is an odd number with $n = 2m+1$. If G has a VMT labelling with magic constant k, then G_J has VMT labelings with magic constants $k_1 = 6m(h_1 + h_2 + \dots + h_l) + k$ and $k_2 = nk - 3m$.*

Given starter cases from Figure 3.1 – 3.4 and Figure 3.12, Theorem 3.6.1 can be used to obtain all our results mentioned in the previous sections. Hence, in this section we presents an example of how Method 1 is modified and used to find a VMT labeling of union of cycles which cannot be obtain using Theorem 3.6.1. Also, in Chapter 4 we show how our methods can also be applied to find EMT (SEMT) labelings for non-regular graphs.

Example: $2C_6 \rightarrow 4C_{12} \cup 2C_6$

Start from a VMT labeling of $2C_6$ given in Figure 3.2. If we use Theorem 3.6.1, we have $G \cong C_6 \cup C_6, h_1 = 6, h_2 = 6$, and $I = \{1, 2\}$. For odd values of n, $J = \emptyset \Rightarrow G_J = C_{6n}$, $J = \{1\} \Rightarrow G_J = nC_6 \cup C_{6n}$ and $J = \{1, 2\} \Rightarrow G_J = \left(nC_6 \cup nC_6 \right) = 2nC_6$. All possible G_J are not congruent with $4C_{12} \cup 2C_6$.

Let κ^+ denote the Kotzig array given in [16] with all its elements increased by 1. Now consider Method 1 using factor $(2r+1) = 5$, with the second column of the Method 1 table are changed to κ^+ instead of κ or κ'. The tables are

Λ	κ^+					Θ_i					Λ	κ^+					Θ_i				
0	1	2	3	4	5	1	2	3	4	5	1	1	2	3	4	5	6	7	8	9	10
14	5	3	1	4	2	75	73	71	74	72	5	5	3	1	4	2	30	28	26	29	27
15	3	4	5	1	2	78	79	80	76	77	23	3	4	5	1	2	118	119	120	116	117
14	1	2	3	4	5	71	72	73	74	75	5	1	2	3	4	5	26	27	28	29	30
4	5	3	1	4	2	25	23	21	24	22	3	5	3	1	4	2	20	18	16	19	17
11	3	4	5	1	2	58	59	60	56	57	21	3	4	5	1	2	108	109	110	106	107
4	1	2	3	4	5	21	22	23	24	25	3	1	2	3	4	5	16	17	18	19	20
13	5	3	1	4	2	70	68	66	69	67	9	5	3	1	4	2	50	48	46	49	47
12	3	4	5	1	2	63	64	65	61	62	17	3	4	5	1	2	88	89	90	86	87
13	1	2	3	4	5	66	67	68	69	70	9	1	2	3	4	5	46	47	48	49	50
6	5	3	1	4	2	35	33	31	34	32	2	5	3	1	4	2	15	13	11	14	12
10	3	4	5	1	2	53	54	51	51	52	18	3	4	5	1	2	93	94	95	91	92
6	1	2	3	4	5	31	32	33	34	35	2	1	2	3	4	5	11	12	13	14	15
7	5	3	1	4	2	40	38	36	39	37	8	5	3	1	4	2	45	43	41	44	42
16	3	4	5	1	2	83	84	85	81	82	19	3	4	5	1	2	98	99	100	96	97
7	1	2	3	4	5	36	37	38	39	40	8	1	2	3	4	5	41	42	43	44	45
0	5	3	1	4	2	5	3	1	4	2	1	5	3	1	4	2	10	8	6	9	7
22	3	4	5	1	2	113	114	115	111	112	20	3	4	5	1	2	103	104	105	101	102

Table 3.3: Tables for $2C_6 \to 4C_{12} \cup 2C_6$

Tracking the numbers from Θ_i in the first table, we have the sequence of edge labels $(1, 75, 22, 68, 31, 40, 2, 73, 21, 70, 32, 38)$ and after that return to 1. These are the edge labels for the first C_{12}.

Starting again still in the first table from $3 = \theta^1_{13}$, we get edge labels for the second C_{12} and starting from $5 = \theta^1_{15}$ we get edge labels for the first C_6.

Performing the same steps in the second table, we have a VMT labeling of $4C_{12} \cup 2C_6$ with $k = 154$ as shown in Figure 3.14.

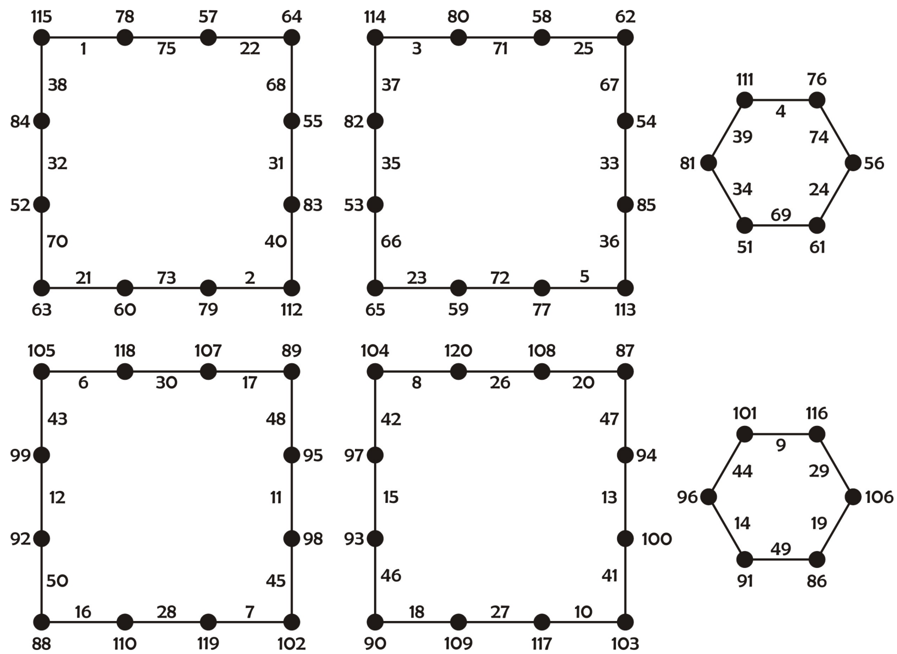


Figure 3.14: VMT labeling for $4C_{12} \cup 2C_6$

Using this method we can come up with VMT (SVMT) labelings which have wider variety of unions of cycles that have not been mentioned in the previous sections. Unfortunately, we have not found a general pattern to determine for which unions of cycles this modified method would work as expected, and leave it for further investigations.

Chapter 4

Results on Edge and Super Edge Magic Total Labelings

In this chapter we modify Method 1 and Method 2 to be applied on EMT (SEMT) labeling instead of VMT (SVMT) labeling. For 2-regular graphs, the results from these methods are similar for VMT and EMT, while that is not the case for other families of graphs. We will discuss how the methods work on the EMT (SEMT) labeling of some families of graphs that are not 2-regular.

4.1 Method 4: Extending the Length of Cycles, Multiplying the Number of Paths

For the family of graphs that contains paths, Method 1 works in a slightly different way. Generally we only switch the order for vertices and edges while the rest is quite the same. As for the result, rather than having the lengths of the paths extended, we have the paths multiplied in number instead.

First make κ, κ' and an $n \times 3$ table as we previously did in Section 3.1. Make a different table for each cycle and for each path contained in the graph.

Change the entry of the first column from $\begin{bmatrix} \lambda(e_i) \\ \lambda(e_{i+1}) \\ \lambda(v_{i+1}) \end{bmatrix}$ to matrix $\Lambda = \begin{bmatrix} \lambda(v_i) \\ \lambda(v_{i+1}) \\ \lambda(e_{i+1}) \end{bmatrix}$.

The rest of the entries are the same as described in Section 3.1.

The entries from the first and second rows of Θ_i will become the labels of adjacent vertices instead of incident edges. The entry from the third row of Θ_i will be the label of the edge between the adjacent vertices.

Suppose a graph G contains C_n and P_m. In the table for the cycle, we track adjacent vertices in each Θ_i, starting from $i = 1$ until $i = n$ and back to $i = 1$ and so on until all numbers in the first two rows in all Θ_i are used.

In the table for paths, if we track adjacent vertices in Θ_i from $i = 1$ until $i = m$, we will not be able to continue to $i = 1$. Instead every time we track adjacent vertices from $i = 1$ until $i = m$, we will get one copy of P_m. Since we have $(2r+1)$ columns in each Θ_i, we end up with $(2r+1)$ copies of P_m instead of $P_{m(2r+1)}$.

4.2 Unions of Cycles and Paths $m\left(C_{n_1(2r+1)} \cup (2r+1)P_{n_2}\right)$

In Section 2.5 we already stated the results given in [3] about SEMT labeling for $C_{n_1} \cup P_{n_2}$. When Method 4 is applied, we have $C_{n_1} \to C_{n_1(2r+1)}$ and $P_{n_2} \to (2r+1)P_{n_2}$.

Modify Method 2 for $C_{n_1} \cup P_{n_2} \to m\,(C_{n_1} \cup P_{n_2})$ as follows.

- Make m copies of labeled $C_{n_1} \cup P_{n_2}$, treat C_{n_1} and P_{n_2} as a pair in each copy.
- Make a $3 \times m$ Kotzig array, multiply each entry of the array by $2(n_1 + n_2) - 1$.
- Add the number in the i^{th} column of the first row to the labels of all vertices with odd subscripts in the i^{th} pair.
- Add the number in the i^{th} column of the second row to the labels of all vertices with even subscripts in the i^{th} pair.
- Add the number in the i^{th} column of the third row to the labels of the edges in the i^{th} pair.

For convenience call this Method 5.

Example: $C_4 \cup CP_2 \rightarrow 3\,(C_4 \cup P_2)$

Start from an SEMT labeling for $C_4 \cup P_2$ as given in [3] is:

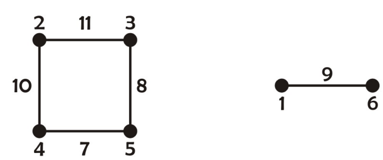


Figure 4.1: SEMT labeling for $C_4 \cup P_2$

The Kotzig array is:

2	1	0
1	0	2
0	2	1

$\xrightarrow{\times 11}$

22	11	0
11	0	22
0	22	11

Figure 4.2: Kotzig array for $C_4 \cup P_2 \rightarrow 3\,(C_4 \cup P_2)$

Adding the number in the Kotzig array to the original labels, we obtain an SEMT labeling for $3\,(C_4 \cup P_2)$ with $k = 49$ as follows.

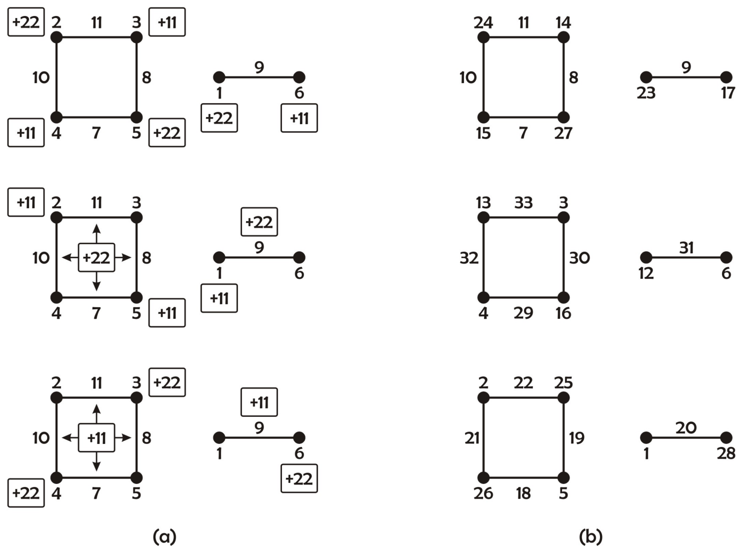


Figure 4.3: (a) Adding process and (b) SEMT labeling for $3\,(C_4 \cup P_2)$

Example: $C_4 \cup CP_2 \to C_{12} \cup 3P_2$

Below is the table used in Method 4 to obtain an SEMT labeling for $C_{12} \cup 3P_2$ from the SEMT labeling of $C_4 \cup P_2$ given in Figure 4.1.

Λ	κ_i or κ'_i			Θ_i		
1	1	2	3	4	5	6
2	2	3	1	8	9	7
10	3	1	2	33	31	32
2	1	2	3	7	8	9
4	2	3	1	14	15	13
7	3	1	2	24	22	23
4	1	2	3	13	14	15
3	2	3	1	11	12	10
6	3	1	2	21	19	20
3	2	3	1	11	12	10
1	1	2	3	4	5	6
9	3	1	2	30	28	29

Λ	κ_i			Θ_i		
0	1	2	3	1	2	3
5	2	3	1	17	18	16
8	3	1	2	27	25	26

Table 4.1: Tables for $C_4 \cup CP_2 \to C_{12} \cup 3P_2$

Figure 4.4 shows the SEMT labeling for $C_{12} \cup 3P_2$ with $k = 45$.

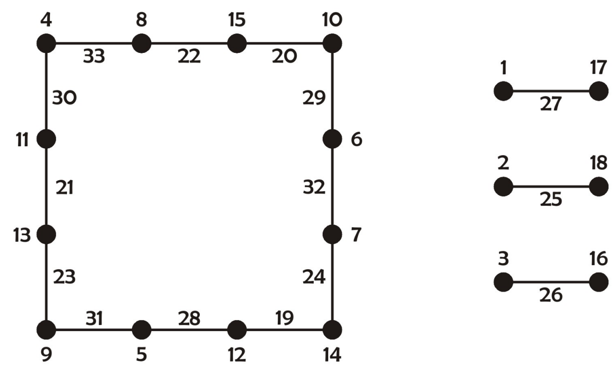


Figure 4.4: SEMT labeling for $C_{12} \cup 3P_2$

Example: $C_4 \cup P_2 \rightarrow C_{20} \cup 5P_2$

The tables are

Λ	κ_i or κ_i'					Θ_i				
1	1	2	3	4	5	6	7	8	9	10
2	3	4	5	1	2	13	14	15	11	12
10	5	3	1	4	2	55	53	51	54	52
2	1	2	3	4	5	11	12	13	14	15
4	3	4	5	1	2	23	24	25	21	22
7	5	3	1	4	2	40	38	36	39	37
4	1	2	3	4	5	21	22	23	24	25
3	3	4	5	1	2	18	19	20	16	17
6	5	3	1	4	2	35	33	31	34	32
3	3	4	5	1	2	18	19	20	16	17
1	1	2	3	4	5	6	7	8	9	10
9	5	3	1	4	2	50	48	46	49	47

Λ	κ_i					Θ_i				
0	1	2	3	4	5	1	2	3	4	5
5	3	4	5	1	2	28	29	30	26	27
8	5	3	1	4	2	45	43	41	44	42

Table 4.2: Tables for $C_4 \cup P_2 \rightarrow C_{20} \cup 5P_2$

Figure 4.5 shows the SEMT labeling for $C_{20} \cup 5P_2$ with $k = 74$.

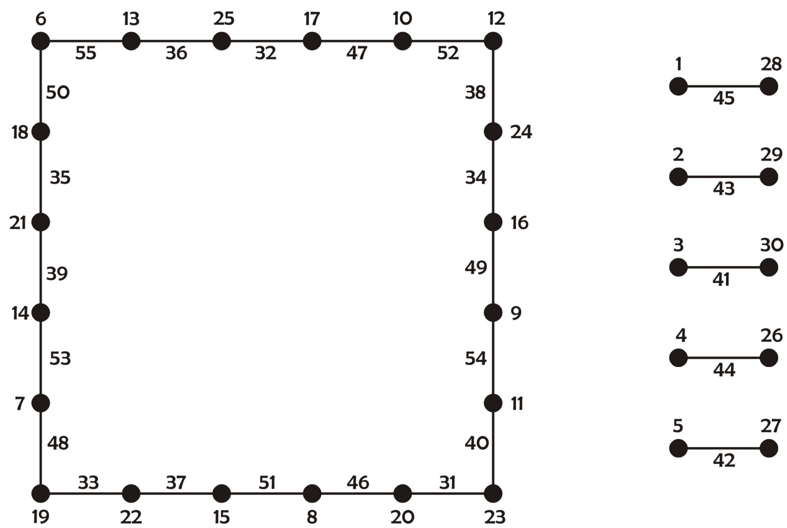


Figure 4.5: SEMT labeling for $C_{20} \cup 5P_2$

Applying both Method 4 and Method 5 to the results listed on Section 2.5 for $C_{n_1} \cup P_{n_2}$, we obtain a several new results.

Theorem 4.2.1. *For any nonnegative integer r, odd m and any integer n_2 with $n_2 \geq 6$, the 2-regular graph $m\left(C_{3(2r+1)} \cup (2r+1)P_{n_2}\right)$ has an SEMT labeling.*

Proof. Applying Method 4 to Theorem 2.5.1, we get an SEMT labeling of $C_{3(2r+1)} \cup (2r+1)P_{n_2}$ for any $r \geq 0$ and $n_2 \geq 6$. Applying Method 5 to this labeling we get an SEMT labeling of $m\left(C_{3(2r+1)} \cup (2r+1)P_{n_2}\right)$ for odd value of m. ■

Theorem 4.2.2. *For any nonnegative integer r and odd values of m, the graph $m\left(C_{4(2r+1)} \cup (2r+1)P_{n_2}\right)$ has an SEMT labeling if $n_2 \neq 3$.*

Proof. Applying Method 4 to Theorem 2.5.2, we get an SEMT labeling of $C_{4(2r+1)} \cup (2r+1)P_{n_2}$ for any $r \geq 0$ and $n_2 \neq 3$. Applying Method 5 to this labeling we get an SEMT labeling of $m\left(C_{4(2r+1)} \cup (2r+1)P_{n_2}\right)$ for odd value of m. ∎

Theorem 4.2.3. *For any nonnegative integer r, odd m and any integer $n_2 \geq 4$, the 2-regular graph $m\left(C_{5(2r+1)} \cup (2r+1)P_{n_2}\right)$ has an SEMT labeling.*

Proof. Applying Method 4 to Theorem 2.5.3, we get an SEMT labeling of $C_{5(2r+1)} \cup (2r+1)P_{n_2}$ for any $r \geq 0$ and $n_2 \geq 4$. Applying Method 5 to this labeling we get an SEMT labeling of $m\left(C_{5(2r+1)} \cup (2r+1)P_{n_2}\right)$ for odd value of m. ∎

Theorem 4.2.4. *For any nonnegative integer r and odd values of m, if n_1 is even with $n_1 \geq 4$ and $n_2 \geq \frac{n_1}{2} + 2$, then the graph $m\left(C_{n_1(2r+1)} \cup (2r+1)P_{n_2}\right)$ has an SEMT labeling.*

Proof. Applying Method 4 to Theorem 2.5.4, we get an SEMT labeling of $C_{n_1(2r+1)} \cup P_{n_2(2r+1)}$ for any $r \geq 0$ and n_1 is even with $n_1 \geq 4$ and $n_2 \geq \frac{n_1}{2} + 2$. Applying Method 5 to this labeling we get an SEMT labeling of $m\left(C_{n_1(2r+1)} \cup P_{n_2(2r+1)}\right)$ for odd value of m. ∎

Corollary 4.2.5. *For any nonnegative integer r, odd m and any positive integer n_2, the graph $m\left(C_{n_1(2r+1)} \cup (2r+1)P_{n_2}\right)$ has an SEMT labeling when $n_1 = 4, 5, 6, 8$ or 10, unless $(n_1, n_2) = (4, 3), (6, 1), (10, 1)$.*

4.3 Cycle with c Chords ${}^{[c]t}C_n$

In this section we apply Method 4 to expand results from Section 2.6.

Theorem 4.3.1. *If the graph ${}^{t}C_n$ has an EMT (SEMT) labeling, then there exists positive integers h and t_h such that for every integer $r \geq 0$, the graph ${}^{[(2r+1)^h]t_h}C_{n(2r+1)^h}$ also has an EMT (SEMT) labeling.*

Proof. Applying Method 4 to an SEMT labeling of ${}^{t}C_n$ give an SEMT labeling of ${}^{[(2r+1)]t_1}C_{n(2r+1)}$ for some value of $t_1 \in \mathbb{N}$. If we apply the method to the SEMT labeling of ${}^{[(2r+1)]t_1}C_{n(2r+1)}$, then we obtain an SEMT labeling of ${}^{[(2r+1)^2]t_2}C_{n(2r+1)^2}$ for some value of $t_2 \in \mathbb{N}$. Performing this n times, we get SEMT labeling for ${}^{[(2r+1)^h]t_h}C_{n(2r+1)^h}$ as stated. ∎

Example: ${}^{2}C_7 \rightarrow {}^{[3]9}C_{21}$

Using the results as listed in Section 2.6, we have an SEMT labeling for ${}^{2}C_7$ with $k = 20$ as shown in Figure 4.6.

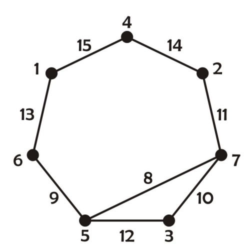


Figure 4.6: SEMT labeling for ${}^{2}C_7$

The table for the chord is

Λ	κ_i			Θ_i		
6	1	2	3	19	20	21
4	2	3	1	14	15	13
7	3	1	2	24	22	23

Table 4.3: Table for ${}^{2}C_7 \rightarrow {}^{[3]9}C_{21}$ (chord)

The table for the cycle is

Λ	κ_i or κ'_i			Θ_i		
0	1	2	3	1	2	3
3	2	3	1	11	12	10
14	3	1	2	45	43	44
3	1	2	3	10	11	12
1	2	3	1	5	6	4
13	3	1	2	42	40	41
1	1	2	3	4	5	6
6	2	3	1	20	21	19
10	3	1	2	33	31	32
6	1	2	3	19	20	21
2	2	3	1	8	9	7
9	3	1	2	30	28	29
2	2	3	1	8	9	7
4	1	2	3	13	14	15
11	3	1	2	36	34	35
4	2	3	1	14	15	13
5	1	2	3	16	17	18
8	3	1	2	27	25	26
5	2	3	1	17	18	16
0	1	2	3	1	2	3
12	3	1	2	39	37	38

Table 4.4: Table for ${}^{2}C_7 \to {}^{[3]9}C_{21}$ (cycle)

From the tables we get an SEMT for ${}^{[3]9}C_{21}$ with $k = 57$ as shown in Figure 4.7.

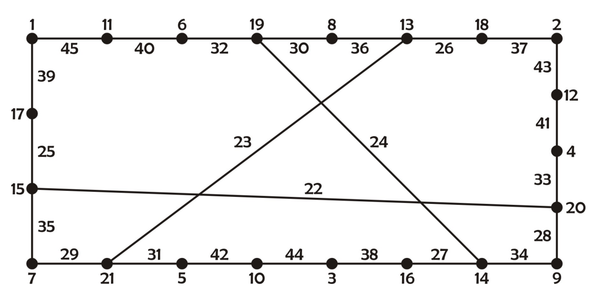


Figure 4.7: SEMT labeling for ${}^{[3]9}C_{21}$

Example: ${}^{2}C_7 \to {}^{[5]9}C_{35}$

The table for the cycle is

Λ	κ_i or κ'_i					Θ_i				
0	1	2	3	4	5	1	2	3	4	5
3	3	4	5	1	2	18	19	20	16	17
14	5	3	1	4	2	75	73	71	74	72
3	1	2	3	4	5	16	17	18	19	20
1	3	4	5	1	2	8	9	10	6	7
13	5	3	1	4	2	70	68	66	69	67
1	1	2	3	4	5	6	7	8	9	10
6	3	4	5	1	2	33	34	35	31	32
10	5	3	1	4	2	55	53	51	54	52
6	1	2	3	4	5	31	32	33	34	35
2	3	4	5	1	2	13	14	15	11	12
9	5	3	1	4	2	50	48	46	49	47
2	3	4	5	1	2	13	14	15	11	12
4	1	2	3	4	5	21	22	23	24	25
11	5	3	1	4	2	60	58	56	59	57
4	3	4	5	1	2	23	24	25	21	22
5	1	2	3	4	5	26	27	28	29	30
8	5	3	1	4	2	45	43	41	44	42
5	3	4	5	1	2	28	29	30	26	27
0	1	2	3	4	5	1	2	3	4	5
12	5	3	1	4	2	65	63	61	64	62

Table 4.5: Table for ${}^{2}C_7 \to {}^{[5]9}C_{35}$ (cycle)

The table for the chord is

Λ	κ_i					Θ_i				
6	1	2	3	4	5	31	32	33	34	35
4	3	4	5	1	2	23	24	25	21	22
7	5	3	1	4	2	40	38	36	39	37

Table 4.6: Table for${}^{2}C_7 \to {}^{[5]9}C_{35}$ (chord)

From the tables we get an SEMT for ${}^{[5]9}C_{35}$ with $k = 94$ as shown in Figure 4.8.

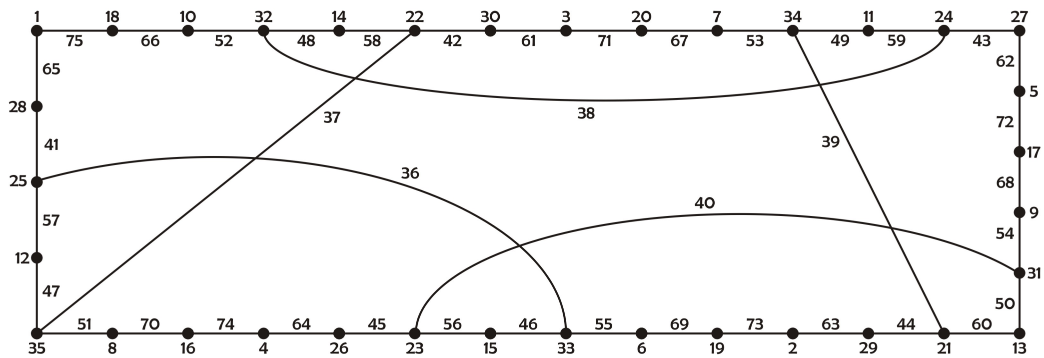


Figure 4.8: SEMT labeling for ${}^{[5]9}C_{35}$

The pattern of how the length of the chord changes in the method is still unknown. However, we do know the location of the chords.

Suppose ${}^{t}C_n$ has a chord connecting the vertices labeled $\lambda(v_a)$ and $\lambda(v_b)$ where $v_a, v_b \in V$. Denote these vertices with the pair notation $(\lambda(v_a), \lambda(v_b))$. Appplying Method 4 to ${}^{t}C_n$, we get ${}^{[2r+1]t_1}C_{(2r+1)n}$. The set of $(2r+1)$ chords written in the pair notation is $\left\{(2r+1)\left(\lambda(v_a)-1\right)+\kappa_{1j},(2r+1)\left(\lambda(v_b)-1\right)+\kappa_{2j}\right\}$.

Observe that if we arrange the second column in the table in such a manner that we have τ^a where $a \neq 1, a \neq 2$ and a is relatively prime to n, then we can obtain different extended graphs. For instance, let us use $\tau^7(\tau^{-1})^0 = \tau^7$ instead of $\tau^4(\tau^{-1})^3 = \tau$ for our example ${}^{2}C_7$.

The table for the chord is the same as Table 4.3.

Λ	κ_i			Θ_i		
6	1	2	3	19	20	21
4	2	3	1	14	15	13
7	3	1	2	24	22	23

Table 4.7: Table for ${}^{2}C_7 \rightarrow {}^{[3]5}C_{21}$ (chord)

Table for the cycle is different in the last three rows from Table 4.4.

Λ	κ_i or κ'_i			Θ_i		
0	1	2	3	1	2	3
3	2	3	1	11	12	10
14	3	1	2	45	43	44
3	1	2	3	10	11	12
1	2	3	1	5	6	4
13	3	1	2	42	40	41
1	1	2	3	4	5	6
6	2	3	1	20	21	19
10	3	1	2	33	31	32
6	1	2	3	19	20	21
2	2	3	1	8	9	7
9	3	1	2	30	28	29
2	1	2	3	7	8	9
4	2	3	1	14	15	13
11	3	1	2	36	34	35
4	1	2	3	13	14	15
5	2	3	1	17	18	16
8	3	1	2	27	25	26
5	1	2	3	16	17	18
0	2	3	1	2	3	1
12	3	1	2	39	37	38

Table 4.8: Table for ${}^{2}C_7 \to {}^{[3]5}C_{21}$ (cycle)

Thus we get ${}^{[3]5}C_{21}$ instead of ${}^{[3]9}C_{21}$ as shown in Figure 4.9.

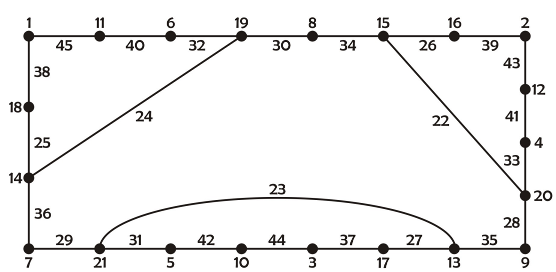


Figure 4.9: SEMT labeling for ${}^{[3]5}C_{21}$

4.4 Lexicographic Product $C_n \circ \overline{K}_2$

An EMT labeling for the complete bipartite graph $K_{4,4}$ as given in [12] is shown in Figure 4.10.

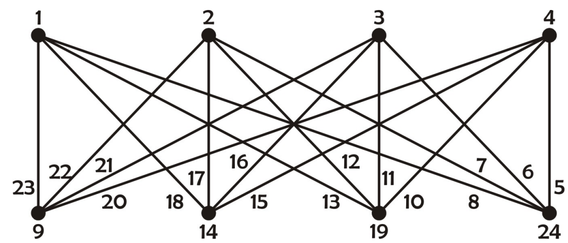


Figure 4.10: EMT labeling for $K_{4,4}$

If we draw it as a lexicographic product $C_4 \circ \overline{K}_2$, we get the next figure.

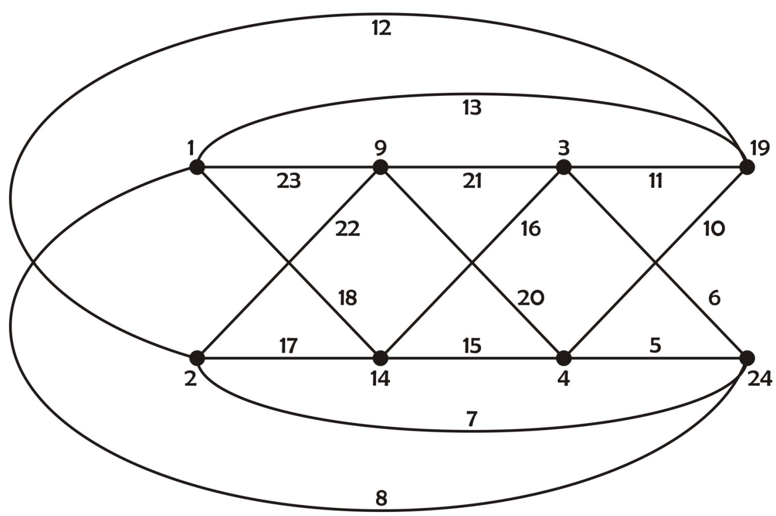


Figure 4.11: EMT labeling for $C_4 \circ \overline{K}_2$

In the following example we will show how to use Method 4 to find an EMT labeling of $C_{4(2r+1)} \circ \overline{K}_2$ from the known EMT labeling of $C_4 \circ \overline{K}_2$.

Example: $C_4 \circ \overline{K}_2 \to C_{12} \circ \overline{K}_2$

First decompose the graph $C_4 \circ \overline{K}_2$ into $4C_4$, where each edge in the edge set of $C_4 \circ \overline{K}_2$ is used exactly once, as shown in Figure 4.12.

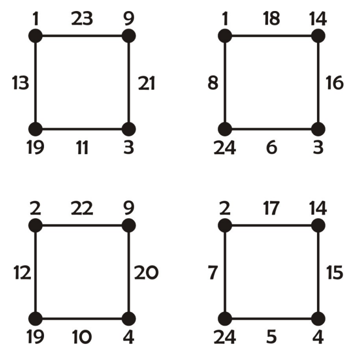


Figure 4.12: Decomposed $C_4 \circ \overline{K}_2$

Apply Method 4 in a similar way to what we did with unions of cycles in Section 3.3. For this example we use $r = 1$ to get $4C_{12}$. The tables for the first two cycles (first row in Figure 4.12) are

Λ	κ_i or κ_i'			Θ_i		
0	1	2	3	1	2	3
8	2	3	1	26	27	25
22	3	1	2	69	67	68
8	1	2	3	25	26	27
2	2	3	1	8	9	7
20	3	1	2	63	61	62
2	1	2	3	7	8	9
18	2	3	1	56	57	55
10	3	1	2	33	31	32
18	2	3	1	56	57	55
0	1	2	3	1	2	3
12	3	1	2	39	37	38

Λ	κ_i or κ_i'			Θ_i		
0	1	2	3	1	2	3
13	2	3	1	41	42	40
17	3	1	2	54	52	53
13	1	2	3	40	41	42
2	2	3	1	8	9	7
15	3	1	2	48	46	47
2	1	2	3	7	8	9
23	2	3	1	71	72	70
5	3	1	2	18	16	17
23	2	3	1	71	72	70
0	1	2	3	1	2	3
7	3	1	2	24	22	23

Table 4.9: Tables for Decomposed $\to$ Extended decomposed $C_4 \circ \overline{K}_2$

The tables for the next two cycles (last row in Figure 4.12) are

Λ	κ_i or κ_i'			Θ_i		
1	1	2	3	4	5	6
8	2	3	1	26	27	25
21	3	1	2	66	64	65
8	1	2	3	25	26	27
3	2	3	1	11	12	10
19	3	1	2	60	58	59
3	1	2	3	10	11	12
18	2	3	1	56	57	55
9	3	1	2	30	28	29
18	2	3	1	56	57	55
1	1	2	3	4	5	6
11	3	1	2	36	34	35

Λ	κ_i or κ_i'			Θ_i		
1	1	2	3	4	5	6
13	2	3	1	41	42	40
16	3	1	2	51	49	50
13	1	2	3	40	41	42
3	2	3	1	11	12	10
14	3	1	2	45	43	44
3	1	2	3	10	11	12
23	2	3	1	71	72	70
4	3	1	2	15	13	14
23	2	3	1	71	72	70
1	1	2	3	4	5	6
6	3	1	2	21	19	20

Table 4.10: Tables for Decomposed $\rightarrow$ Extended decomposed $C_4 \circ \overline{K}_2$

Hence we have $4C_{12}$ as shown in Figure 4.13.

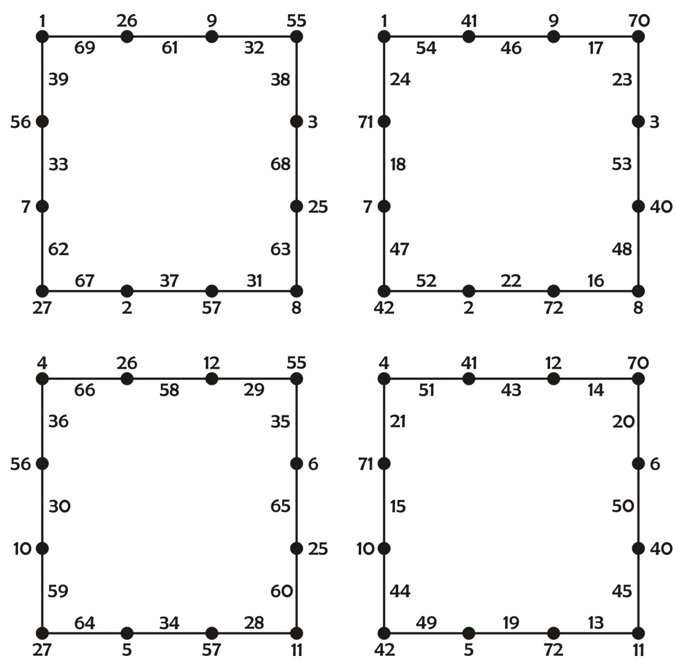


Figure 4.13: Extended decomposed $C_4 \circ \overline{K}_2$

Then we combine these cycles to get $C_{12} \circ \overline{K}_2$ as shown in Figure 4.14.

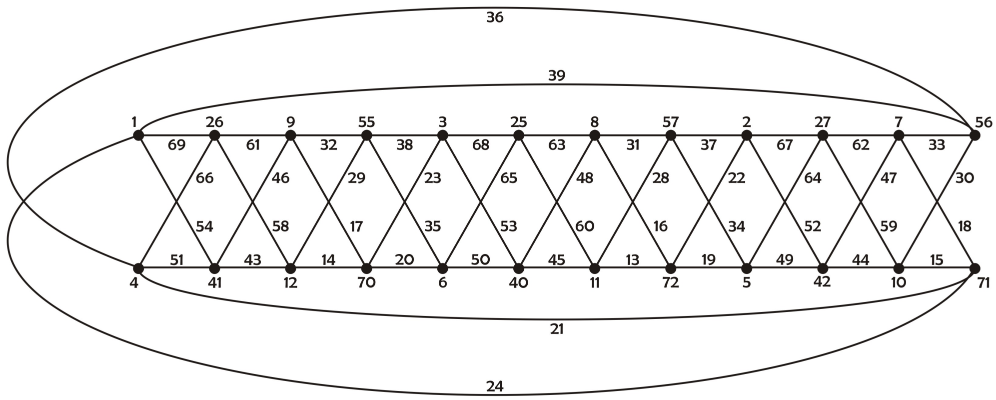


Figure 4.14: EMT labeling for $C_{12} \circ \overline{K}_2$

Now we modify Method 2 to find an EMT labeling of $mC_4 \circ \overline{K}_2$ from the known EMT labeling of $C_4 \circ \overline{K}_2$. For convenience let us call this Method 6.

Modify the methods in Section 3.2 for $C_n \circ \overline{K}_2 \to m\left(C_n \circ \overline{K}_2\right)$ as follows.

- Make m copies of a labeled decomposed $C_n \circ \overline{K}_2$, that is, m copies of $4C_n$, and treat the copies as groups of four.
- Make a $3 \times m$ Kotzig array, multiply each entry of the array by $6n$.
- Add the number in the i^{th} column of the first row to the labels of all vertices with odd subscripts in the i^{th} group.
- Add the number in the i^{th} column of the second row to the labels of all vertices with even subscripts in the i^{th} group.
- Add the number in the i^{th} column of the third row to the labels of the edges in the i^{th} group.

In the next example we use Method 6 to find an EMT labeling of $3\left(C_4 \circ \overline{K}_2\right)$.

Example: $C_4 \circ \overline{K}_2 \to 3\left(C_4 \circ \overline{K}_2\right)$

Adding the numbers from the Kotzig array to the 3 copies of $4C_4$, we have the following.

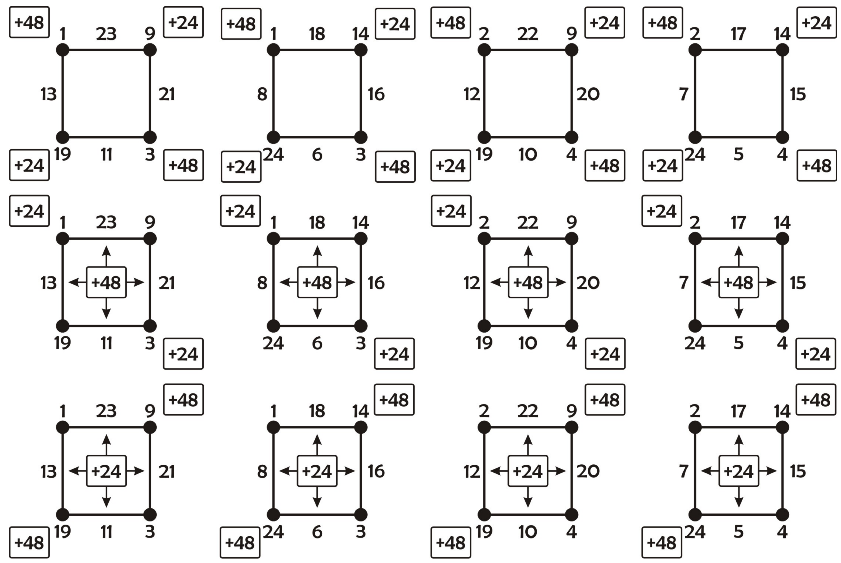


Figure 4.15: Adding the numbers from the array

Hence we get the labeled $12C_4$ as follow.

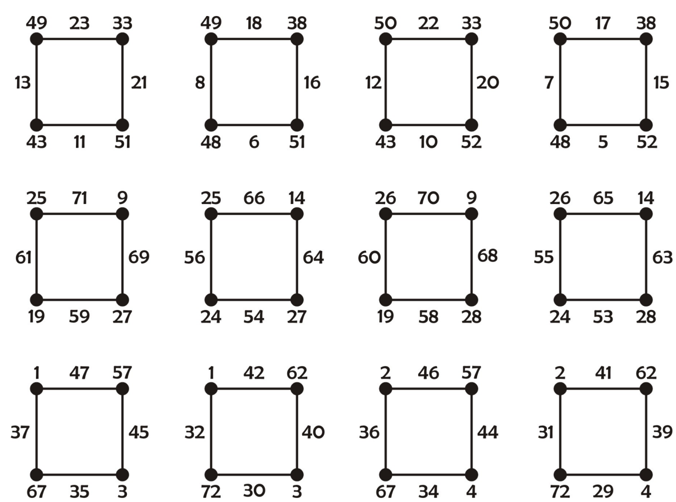


Figure 4.16: Labeling for $3(4C_4)$

Combining these 12 cycles, we get an EMT labeling of $3\left(C_4 \circ \overline{K}_2\right)$ as shown in Figure 4.17.

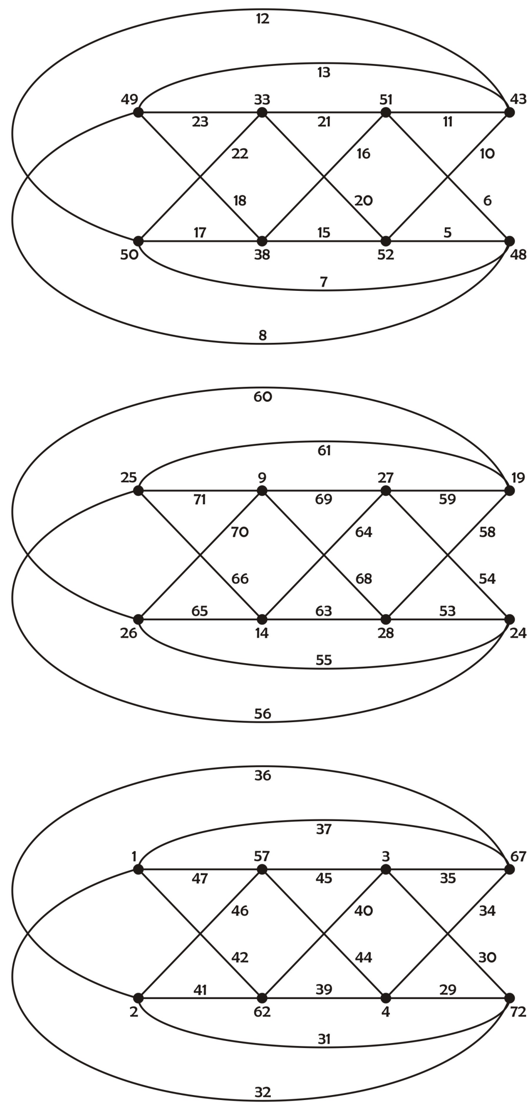


Figure 4.17: EMT labeling for $3\left(C_4 \circ \overline{K}_2\right)$

Observe that our methods work on this lexicographic product since we can decompose the graph into separate cycles with every edge used exactly once, so that we can treat them as unions of cycles and apply the methods accordingly.

Therefore we can summarize our result in the next theorem.

Theorem 4.4.1. *For $r \geq 0$, odd values of m and $n = 4(2r+1)$, the graph $m\left(C_n \circ \overline{K}_2\right)$ has an EMT labeling.*

Proof. Applying Method 4 to Theorem 2.7.1, we get an SVMT labeling of $C_n \circ \overline{K}_2$ for $n = 4(2r+1), r \geq 0$. Applying Method 6 to this labeling we get an SVMT labeling of $m\left(C_n \circ \overline{K}_2\right)$ for an odd value of m. ∎

4.5 Cartesian Product $P_2 \,\square\, C_n$

In this section we give an example of how to use Method 4 to provide alternative ways of constructing EMT (SEMT) labelings of the Cartesian product $P_2 \,\square\, C_n$. Using this method we can use the known EMT (SEMT) labeling of $P_m \,\square\, C_n$ to obtain an EMT (SEMT) labeling of $P_m \,\square\, C_{n(2r+1)}$.

Example: $P_2 \,\square\, C_3 \to P_2 \,\square\, C_9$

Using the results as listed in Section 2.8, we have an SEMT labeling for $P_2 \,\square\, C_3$ with $k = 18$ as shown in Figure 4.18.

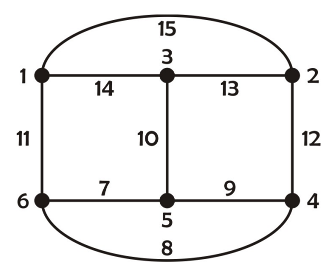


Figure 4.18: SEMT labeling for $P_2 \,\square\, C_3$

The tables are given below.

Λ	κ_i			Θ_i		
0	1	2	3	1	2	3
5	2	3	1	17	18	16
10	3	1	2	33	31	32

Λ	κ_i			Θ_i		
2	1	2	3	7	8	9
4	2	3	1	14	15	13
9	3	1	2	30	28	29

Λ	κ_i			Θ_i		
1	1	2	3	4	5	6
3	2	3	1	11	12	10
11	3	1	2	36	34	35

Table 4.11: Tables for $P_2 \square C_3 \to P_2 \square C_9$ (vertical paths)

Λ	κ_i or κ'_i			Θ_i		
0	1	2	3	1	2	3
2	2	3	1	8	9	7
13	3	1	2	42	40	41
2	1	2	3	7	8	9
1	2	3	1	5	6	4
12	3	1	2	39	37	38
1	2	3	1	5	6	4
0	1	2	3	1	2	3
14	3	1	2	45	43	44

Λ	κ_i or κ'_i			Θ_i		
5	1	2	3	16	17	18
4	2	3	1	14	15	13
6	3	1	2	21	19	20
4	1	2	3	13	14	15
3	2	3	1	11	12	10
8	3	1	2	27	25	26
3	2	3	1	11	12	10
5	1	2	3	16	17	18
7	3	1	2	24	22	23

Table 4.12: Tables for $P_2 \square C_3 \to P_2 \square C_9$ (cycles)

From the tables we get an SEMT for $P_2 \square C_9$ with $k = 51$ as shown in Figure 4.19.

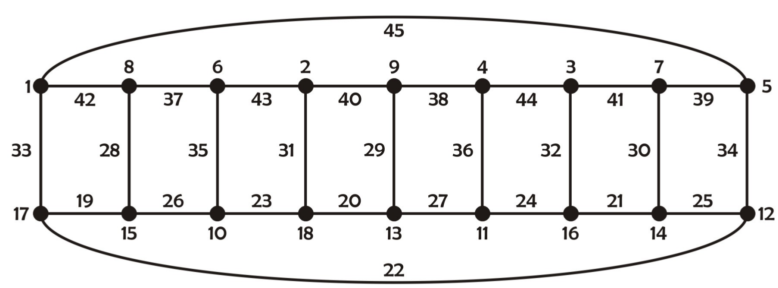


Figure 4.19: SEMT labeling for $P_2 \square C_9$

As mentioned in Section 2.8, all results for EMT labelings of $P_m \square C_n$ for odd values of n are already published, so what is stated in this section is only an alternative way to find such EMT labelings of $P_m \square C_{n(2r+1)}$ from the known labelings of $P_m \square C_n$ for any positive r.

4.6 Cartesian Product $P_2 \square P_n$

Unsurprisingly, when we apply Method 4 to the graph $P_2 \square P_n$, it will multiply the number of the ladders instead of extending its length. From the known SEMT labeling of $P_2 \square P_n$ as listed in Section 2.9, we can obtain the SEMT labeling of $m\,(P_2 \square P_n)$ for any odd values of m as we explain in the following example.

Example: $P_2 \square P_5 \to m\,(P_2 \square P_5)$

In [26], an SEMT labeling of $P_2 \square P_5$ with $k = 28$ is given as shown in Figure 4.20.

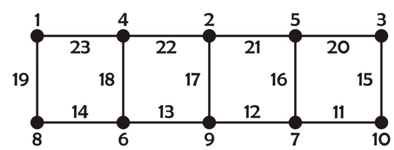


Figure 4.20: SEMT labeling for $P_2 \square P_5$

Tables for two P_5 are

Λ	κ_i or κ'_i			Θ_i		
0	1	2	3	1	2	3
3	2	3	1	11	12	10
22	3	1	2	69	67	68
3	1	2	3	10	11	12
1	2	3	1	5	6	4
21	3	1	2	66	64	65
1	1	2	3	4	5	6
4	2	3	1	14	15	13
20	3	1	2	63	61	62
4	2	3	1	14	15	13
2	1	2	3	7	8	9
19	3	1	2	60	58	59

Λ	κ_i or κ'_i			Θ_i		
7	1	2	3	22	23	24
5	2	3	1	17	18	16
13	3	1	2	42	40	41
5	1	2	3	16	17	18
8	2	3	1	26	27	25
12	3	1	2	39	37	38
8	1	2	3	25	26	27
6	2	3	1	20	21	19
11	3	1	2	36	34	35
6	2	3	1	20	21	19
9	1	2	3	28	29	30
10	3	1	2	33	31	32

Table 4.13: Tables for $P_2 \square P_5 \to m\,(P_2 \square P_5)$ for P_5

Tables for five P_2 are

Λ	κ_i			Θ_i		
0	1	2	3	1	2	3
7	2	3	1	23	24	22
18	3	1	2	57	55	56

Λ	κ_i			Θ_i		
3	1	2	3	10	11	12
5	2	3	1	17	18	16
17	3	1	2	54	52	53

Λ	κ_i			Θ_i		
1	1	2	3	4	5	6
8	2	3	1	26	27	25
16	3	1	2	51	49	50

Λ	κ_i			Θ_i		
4	1	2	3	13	14	15
6	2	3	1	20	21	19
15	3	1	2	48	46	47

Λ	κ_i			Θ_i		
2	1	2	3	7	8	9
9	2	3	1	29	30	28
14	3	1	2	45	43	44

Table 4.14: Tables for $P_2 \square P_5 \to m\,(P_2 \square P_5)$ for P_2

From the tables we get SEMT for $P_2 \square P_5$ with $k = 81$ as shown in Figure 4.21.

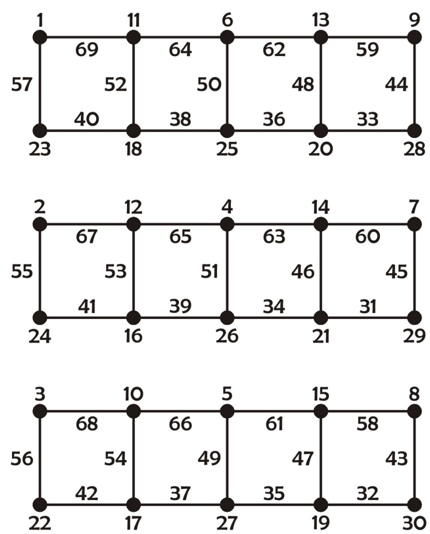


Figure 4.21: SEMT labeling for $3(P_2 \square P_5)$

Hence we can summarize our new result in Theorem 4.6.1.

Theorem 4.6.1. *For any odd values of m and n, the graph $m\,(P_2 \square P_n)$ has an SEMT labeling with $k = \frac{m}{2}\,(11n - 1)$.*

Proof. The result follows from applying Method 4 to result from Theorem 2.9.2. ∎

4.7 Unions of Paths mP_n

In this section we apply Method 4 to the results listed on Section 2.10. For unions of paths $P_m \cup P_n$ and $P_3 \cup mP_2$, the method will multiply the number of graphs by a factor of an odd number $2r+1$. Details are shown in the next two examples.

Example: $P_4 \cup P_2 \rightarrow 3(P_4 \cup P_2)$

In [3], an SEMT labeling of $P_4 \cup P_2$ with $k = 16$ is given as shown in Figure 4.22.

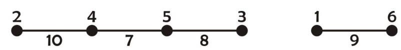


Figure 4.22: SEMT labeling for $P_4 \cup P_2$

Tables for the method are

Λ	κ_i or κ_i'			Θ_i		
1	1	2	3	4	5	6
3	2	3	1	11	12	10
9	3	1	2	30	28	29
3	1	2	3	10	11	12
4	2	3	1	14	15	13
6	3	1	2	21	19	20
4	2	3	1	13	14	15
2	1	2	3	7	8	9
7	3	1	2	24	22	23

Λ	κ_i			Θ_i		
6	1	2	3	1	2	3
4	2	3	1	17	18	16
7	3	1	2	27	25	26

Table 4.15: Tables for $P_4 \cup P_2 \rightarrow 3(P_4 \cup P_2)$

From the tables we get an SEMT for $3(P_4 \cup P_2)$ with $k = 45$ as shown in Figure 4.23.

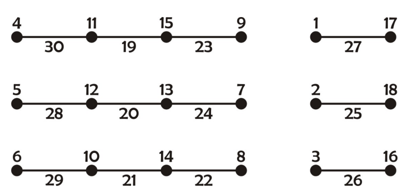


Figure 4.23: SEMT labeling for $3(P_4 \cup P_2)$

Example: $P_3 \cup 3P_2 \to 5(P_3 \cup 3P_2)$

An SEMT labeling of $P_3 \cup 3P_2$ with $k = 22$ is given as shown in Figure 4.24.

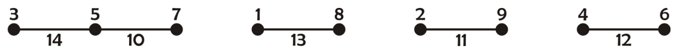


Figure 4.24: SEMT labeling for $P_3 \cup 3P_2$

Tables for the method are

Λ	κ_i or κ_i'					Θ_i				
2	1	2	3	4	5	11	12	13	14	15
4	3	4	5	1	2	23	24	25	21	22
13	5	3	1	4	2	70	68	66	69	67
4	1	2	3	4	5	21	22	23	24	25
6	3	4	5	1	2	33	34	35	31	32
9	5	3	1	4	2	50	48	46	49	47

Λ	κ_i					Θ_i				
0	1	2	3	4	5	1	2	3	4	5
7	3	4	5	1	2	38	39	40	36	37
12	5	3	1	4	2	65	63	61	64	62

Λ	κ_i					Θ_i				
1	1	2	3	4	5	6	7	8	9	10
8	3	4	5	1	2	43	44	45	41	42
10	5	3	1	4	2	55	53	51	54	52

Λ	κ_i					Θ_i				
3	1	2	3	4	5	16	17	18	19	20
5	3	4	5	1	2	28	29	30	26	27
11	5	3	1	4	2	60	58	56	59	57

Table 4.16: Tables for $P_3 \cup 3P_2 \to 5(P_3 \cup 3P_2)$

From the tables we get an SEMT for $5(P_3 \cup 3P_2)$ with $k = 104$ as shown in Figure 4.25.

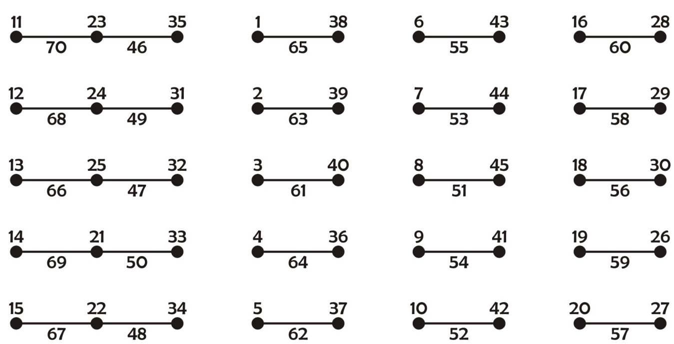


Figure 4.25: SEMT labeling for $5(P_3 \cup 3P_2)$

From the known SEMT labelings of $2P_n$, we can obtain SEMT labelings of $2(2r+1)P_n$ for any positive value of r. Details are given in the follwoing example.

Example: $2P_4 \to 6P_4$

In [5], an SEMT labeling of $2P_4$ with $k = 21$ was given as shown in Figure 4.26.

Figure 4.26: SEMT labeling for $2P_4$

Tables for two P_4 are.

Λ	κ_i or κ'_i			Θ_i		
0	1	2	3	1	2	3
3	2	3	1	11	12	10
22	3	1	2	69	67	68
3	1	2	3	10	11	12
1	2	3	1	5	6	4
21	3	1	2	66	64	65
1	1	2	3	4	5	6
4	2	3	1	14	15	13
20	3	1	2	63	61	62
4	2	3	1	14	15	13
2	1	2	3	7	8	9
19	3	1	2	60	58	59

Λ	κ_i or κ'_i			Θ_i		
7	1	2	3	22	23	24
5	2	3	1	17	18	16
13	3	1	2	42	40	41
5	1	2	3	16	17	18
8	2	3	1	26	27	25
12	3	1	2	39	37	38
8	1	2	3	25	26	27
6	2	3	1	20	21	19
11	3	1	2	36	34	35
6	2	3	1	20	21	19
9	1	2	3	28	29	30
10	3	1	2	33	31	32

Table 4.17: Tables for $2P_4 \to 6P_4$

From the tables we get an SEMT for $6P_4$ with $k = 60$ as shown in Figure 4.27.

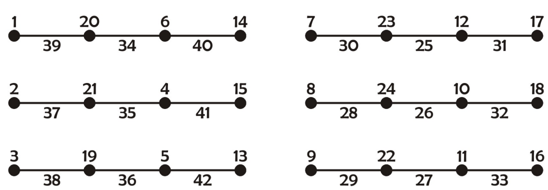


Figure 4.27: SEMT labeling for $6P_4$

We can summarize our new results in Theorems 4.7.1 – 4.7.4.

Theorem 4.7.1. *If* $(m, n) \neq (2, 2)$ *or* $(3, 3)$*, then the union of paths* $(2r+1)(P_m \cup P_n)$ *has an SEMT labeling for all nonnegative integers* r*.*

Proof. The result follows from applying Method 4 to the graph in Theorem 2.10.1. ■

Theorem 4.7.2. *The graph* $(2r+1)(P_3 \cup mP_2)$ *has an SEMT labeling for all nonnegative integers* m *and* r*.*

Proof. The result follows from applying Method 4 to the graph in Theorem 2.10.3. ■

Theorem 4.7.3. *For an even value of* m*,* $m \equiv 2 \pmod 4$*, the graph* mP_n *has an SEMT labeling if* n *is not equal to* 2 *or* 3*.*

Proof. The result follows from applying Method 4 to the graph in Theorem 2.10.5. ■

Theorem 4.7.4. *For all* $n \geq 2$*, even value of* m*,* $m \equiv 2 \pmod 4$*, the graph* mP_{4n} *has an SEMT labeling.*

Proof. The result follows from applying Method 4 to the result from Theorem 2.10.6. ■

4.8 (n, t)-tadpole and Mutated (m, n, t)-tadpole

Before we apply our Methods to the results stated in Section 2.11, we will first state our new result on tadpole graph.

Theorem 4.8.1. *A* $(3, t)$*-tadpole graph has an EMT labeling with magic constant* $k = \frac{1}{2}(5t + 20)$ *when* t *is even and with* $k = \frac{1}{2}(5t + 21)$ *when* t *is odd.*

Proof. For an even t, define

$$\lambda(v_i) = \begin{cases} t+3 & \text{for } i=1 \\ \frac{1}{2}(t+2) & \text{for } i=2 \\ t+5 & \text{for } i=3 \end{cases} \qquad \lambda(u_i) = \begin{cases} \frac{1}{2}(t-i+1) & \text{for } i \text{ odd} \\ \frac{1}{2}(4t-i-6) & \text{for } i \text{ even} \end{cases}$$

For an odd t, define

$$\lambda(v_i) = \begin{cases} t+3 & \text{for } i=1 \\ \frac{1}{2}(t+3) & \text{for } i=2 \\ t+5 & \text{for } i=3 \end{cases} \qquad \lambda(u_i) = \begin{cases} \frac{1}{2}(t-i+2) & \text{for } i \text{ odd} \\ t+3-\frac{i}{2} & \text{for } i \text{ even} \end{cases}$$

The labels of the edges are forced by the value of k. ■

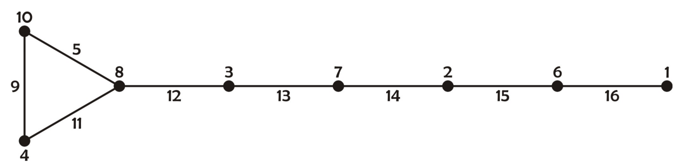


Figure 4.28: EMT labeling for $(3,5)$-tadpole with $k=23$.

Theorem 4.8.2. *A $(4,t)$-tadpole graph has an EMT labeling with magic constant $k = \frac{1}{2}(5t+23)$ when t is odd and with $k = \frac{1}{2}(5t+24)$ when t is even.*

Proof. Define

$$\lambda(v_i) = \begin{cases} \frac{1}{2}(t+i+1) & \text{for } i \text{ odd, } t \text{ even} \\ \frac{1}{2}(t+i) & \text{for } i \text{ odd, } t \text{ odd} \\ \frac{1}{2}(2t+3i) & \text{for } i \text{ even} \end{cases}$$

$$\lambda(u_i) = \begin{cases} \frac{1}{2}(2t-i+5) & \text{for } i \text{ odd} \\ \frac{1}{2}(t-i+2) & \text{for } i \text{ even and } t \text{ even} \\ \frac{1}{2}(t-i+1) & \text{for } i \text{ even and } t \text{ odd} \end{cases}$$

The labels of the edges are forced by the value of k. ■

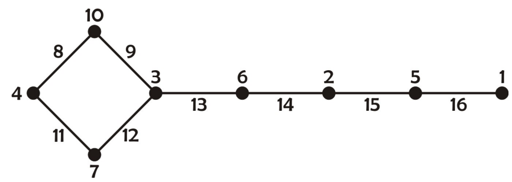


Figure 4.29: EMT labeling for $(4,4)$-tadpole with $k = 22$.

In [18] it has been proved that an $(n,2)$-tadpole graph has an SEMT labeling if and only if n is even. Theorem 4.8.3 gives a construction of an EMT labeling for an $(n,2)$-tadpole graph when n is odd.

Theorem 4.8.3. *An $(n,2)$-tadpole graph has an EMT labeling with magic constant $k = \frac{1}{2}(5n+13)$ when n is odd.*

Proof. Define

$$\lambda(v_i) = \begin{cases} n & \text{for } i = 1 \\ \frac{1}{2}(n-1) & \text{for } i = 2 \\ n+3 & \text{for } i = 3 \\ \frac{1}{2}(n+i-1) & \text{for } i = 4, 6, ..., n-1 \\ \frac{1}{2}(i-3) & \text{for } i = 5, 7, ..., n \end{cases}$$

$$\lambda(u_i) = \begin{cases} \frac{1}{2}(n+1) & \text{for } i = 1 \\ (n+1) & \text{for } i = 2 \end{cases}$$

The labels of the edges are forced by the value of k. ■

Figure 4.30 shows an EMT labeling for $(11,2)$-tadpole graph.

Theorem 4.8.4 gives a construction for an EMT labeling of an $(n,1)$-tadpole graph when n is even, with a different magic constant from what found in [25].

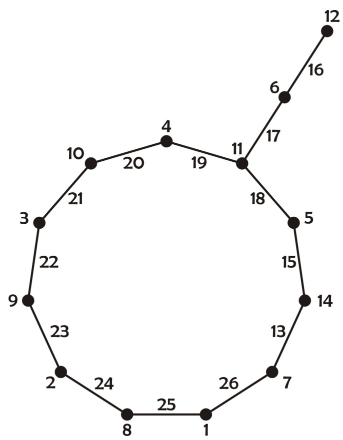


Figure 4.30: EMT labeling for $(11,2)$-tadpole graph.

Theorem 4.8.4. *An* $(n,1)$*-tadpole graph has an EMT labeling with magic constant* $k = \frac{1}{2}(7n+2)$ *when* n *is even.*

Proof. Let v_0v_n be the tail of the tadpole.

For $n \equiv 0 \pmod 4$, define

$$\lambda(v_i) = \begin{cases} n & \text{for } i = 0 \\ \frac{n}{2} - 1 & \text{for } i = 1 \\ n+1 & \text{for } i = 2 \\ 2n+1-\frac{i+1}{2} & \text{for } i = 3, 5, ..., \frac{n}{2}+1 \\ \frac{3n+6-i}{2} & \text{for } i = 4, 6, ..., \frac{n}{2} \\ \frac{3n}{4} - 1 + i & \text{for } i = \frac{n}{2}+2, \frac{n}{2}+3 \\ \frac{5n}{4} & \text{for } i = \frac{n}{2}+4 \\ \frac{3n-i+4}{2} & \text{for } i = \frac{n}{2}+6, \frac{n}{2}+8, ..., n-2 \\ \frac{3n+i-7}{2} & \text{for } i = \frac{n}{2}+5, \frac{n}{2}+7, ..., n-1 \\ n+2 & \text{for } i = n \end{cases}$$

For $n \equiv 2 \pmod 4$, define

$$\lambda\left(v_i\right) = \begin{cases} n & \text{for } i = 0 \\ \frac{n}{2} - 1 & \text{for } i = 1 \\ n + 1 & \text{for } i = 2 \\ 2n + 1 - \frac{i+1}{2} & \text{for } i = 3, 5, ..., \frac{n}{2} \\ \frac{3n+6-i}{2} & \text{for } i = 4, 6, ..., \frac{n}{2} + 3 \\ \frac{3n-i+3}{2} & \text{for } i = \frac{n}{2} + 2 \\ \frac{3n-i+4}{2} & \text{for } i = \frac{n}{2} + 4, \frac{n}{2} + 6, ..., n - 2 \\ 2n - \frac{i-3}{2} & \text{for } i = \frac{n}{2} + 5, \frac{n}{2} + 7, ..., n - 1 \\ n + 2 & \text{for } i = n \end{cases}$$

The labels of the edges are forced by the value of k. ■

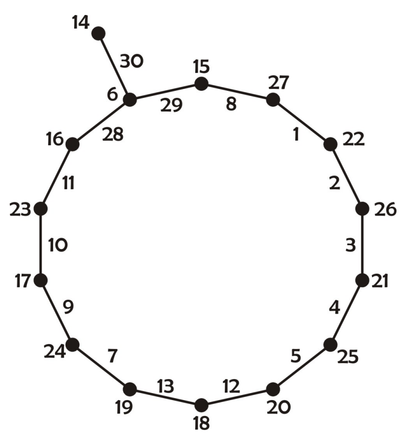


Figure 4.31: EMT labeling for $(14, 1)$-tadpole graph.

If we apply Method 4 to all results in Section 2.11 and also the results mentioned above in this section, we will obtain results for the mutated tadpoles.

Example: $(4, 2)$**-tadpole** $\rightarrow$ **mutated** $(3, 12, 2)$**-tadpole**

In [18], an SEMT labeling of $(4, 2)$-tadpole with $k = 17$ given as shown in Figure 4.32.

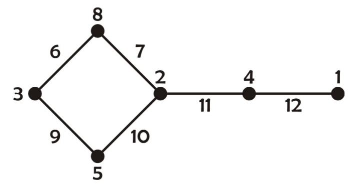


Figure 4.32: An SEMT labeling for $(4, 2)$-tadpole

The tables are given below.

Λ	κ_i or κ_i'			Θ_i		
2	1	2	3	7	8	9
7	2	3	1	23	24	22
5	3	1	2	18	16	17
7	1	2	3	22	23	24
1	2	3	1	5	6	4
6	3	1	2	21	19	20
1	1	2	3	4	5	6
4	2	3	1	14	15	13
9	3	1	2	30	28	29
4	2	3	1	14	15	13
2	1	2	3	7	8	9
8	3	1	2	27	25	26

Table 4.18: Table for $(4, 2)$-tadpole $\rightarrow$ mutated $(3, 12, 2)$-tadpole (body)

Λ	κ_i			Θ_i		
1	1	2	3	4	5	6
3	2	3	1	11	12	10
10	3	1	2	33	31	32
3	1	2	3	10	11	12
0	2	3	1	2	3	1
11	3	1	2	36	34	35

Table 4.19: Table for $(4, 2)$-tadpole $\rightarrow$ mutated $(3, 12, 2)$-tadpole (tail)

From the tables we get an SEMT for mutated $(3, 12, 2)$-tadpole with $k = 48$ as shown in Figure 4.33.

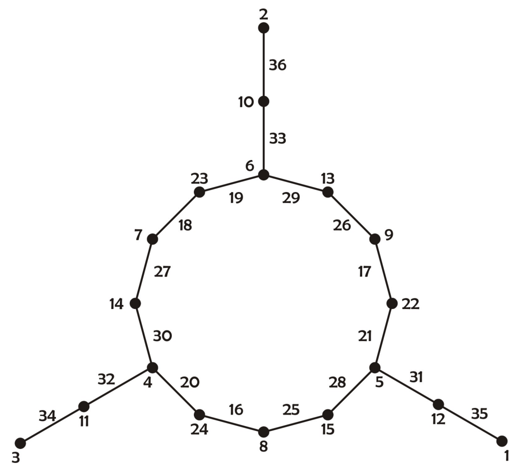


Figure 4.33: An SEMT labeling for mutated $(3, 12, 2)$-tadpole

We can summarize our results for mutated tadpole graphs in the following theorems.

Theorem 4.8.5. *For any integers $n, r \geq 0$, the mutated $((2r+1), (2r+1)n, 1))$-tadpole has an SEMT labeling when n is odd and an EMT labeling when n is even.*

Proof. By applying Method 4 to the result in Theorem 2.11.2 we get an SEMT labeling for mutated $((2r+1), (2r+1)n, 1))$-tadpole when n is odd. Applying it to the result in Theorem 2.11.1 by a factor of $(2r+1)$ we get an EMT labeling for mutated $((2r+1), (2r+1)n, 1))$-tadpole when n is even. ■

Theorem 4.8.6. *For any integers $r \geq 0$, the mutated $((2r+1), (2r+1)n, 2))$-tadpole has an SEMT labeling when n is even and an EMT labeling when n is odd.*

Proof. By applying Method 4 to the result in Theorem 2.11.3 we get an SEMT labeling for mutated $((2r+1), (2r+1)n, 2))$-tadpole when n is even. Applying it to the result in Theorem 4.8.3 by a factor of $(2r+1)$ we get an EMT labeling for mutated $((2r+1), (2r+1)n, 2))$-tadpole when n is odd. ■

Theorem 4.8.7. *For any integers $t, r \geq 0$, the mutated $((2r+1), 3(2r+1), t))$-tadpole has an EMT labeling.*

Proof. By applying Method 4 to the result in Theorem 4.8.1 by a factor of $(2r+1)$ we get an EMT labeling for mutated $((2r+1), 3(2r+1), t))$-tadpole. ■

Theorem 4.8.8. *For any integers $t, r \geq 0$, the mutated $((2r+1), 4(2r+1), t))$-tadpole has an EMT labeling.*

Proof. By applying Method 4 to the result in Theorem 4.8.2 by a factor of $(2r+1)$ we get an EMT labeling for mutated $((2r+1), 4(2r+1), t))$-tadpole. ■

4.9 Method 4 Performed on Friendship Graph Fr_n

By applying Method 4 to SEMT labelings of friendship graphs, we obtain SEMT labeling for a new family of graph. To see how the method works we include the example below:

Example: From Fr_3 using factor $(2r+1) = 3$

In [22], SEMT labeling of Fr_3 with $k = 20$ given as shown in Figure 4.34.

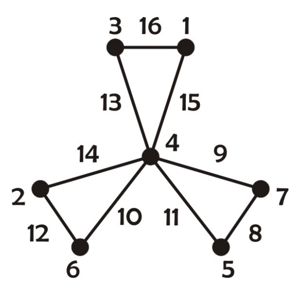


Figure 4.34: SEMT labeling for Fr_3

For the tables we treat each triangles in Fr_n as a cycle C_3, and make separate table for each cycle.

Λ	κ_i or κ_i'			Θ_i			Λ	κ_i or κ_i'			Θ_i			Λ	κ_i or κ_i'			Θ_i		
3	1	2	3	10	11	12	3	1	2	3	10	11	12	3	1	2	3	10	11	12
0	2	3	1	2	3	1	6	2	3	1	20	21	19	5	2	3	1	17	18	16
14	3	1	2	45	43	44	8	3	1	2	27	25	26	9	3	1	2	30	28	29
0	1	2	3	1	2	3	6	1	2	3	19	20	21	5	1	2	3	16	17	18
2	2	3	1	8	9	7	4	2	3	1	14	15	13	1	2	3	1	5	6	4
15	3	1	2	48	46	47	7	3	1	2	24	22	23	11	3	1	2	36	34	35
2	2	3	1	8	9	7	4	2	3	1	14	15	13	1	2	3	1	5	6	4
3	1	2	3	10	11	12	3	1	2	3	10	11	12	3	1	2	3	10	11	12
12	3	1	2	39	37	38	10	3	1	2	33	31	32	13	3	1	2	42	40	41

Table 4.20: Tables for Method 4 performed on Fr_3 with factor $(2r+1)=3$

From the tables we get an SEMT for a new graph that shown in Figure 4.35.

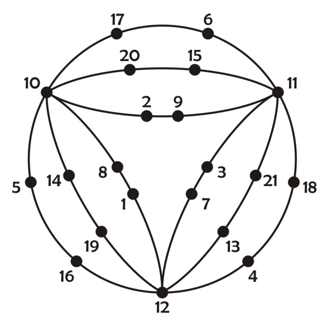


Figure 4.35: SEMT labeling for new graph from Fr_3

Due to limited space in the graph, the edge labels are not included in the figure. They can be found in the tables if required.

For convenience let us denote this resulting graph by $C_t(n,m)$, where t is the number of vertices in each cycle, m is the number of common vertices where the distance between common vertices is always 2 and n is the number of triangles in the original friendship graph, which will become the number of layers of cycles (from inner to outer cycles) in the resulting graph.

This way the graph in Figure 4.35 is denoted as $C_9(3,3)$. It has 3 layers of cycles (inner, middle and outer cycle), that do not share any common edges.

Theorem 4.9.1. *The graph $C_{3m}(n, m)$ has an SEMT labeling when m is odd and $n \in \{3, 4, 5, 7\}$.*

Proof. Observe that by applying Method 4 to a friendship graph using factor $m = 2r + 1$, every triangle in Fr_n will become a cycle of length $3m$, so the number of triangles (n) will become the number of layers in the new graph. Hence from SEMT labelings of all feasible values of n for friendship graph Fr_n that are stated in Theorem 2.12.1, we get the result above. ■

4.10 Method 4 Performed on Fan f_n

Example: From f_4 using factor $(2r + 1) = 3$

In [22], EMT labeling of f_4 with $k = 18$ given as shown in Figure 4.36.

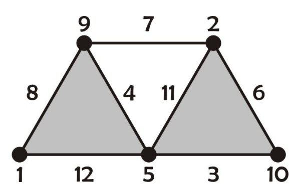


Figure 4.36: EMT labeling for f_4

In performing Method 4 to this graph, treat the shaded area as cycles, so we have 2 cycles and a path. The tables for the path is

Λ	κ_i			Θ_i		
8	1	2	3	25	26	27
1	2	3	1	5	6	4
6	3	1	2	21	19	20

Table 4.21: Table for Method 4 performed on f_4 with factor $(2r + 1) = 3$ (path)

The table for the cycles are

Λ	κ_i or κ'_i			Θ_i		
0	1	2	3	1	2	3
8	2	3	1	26	27	25
7	3	1	2	24	22	23
8	1	2	3	25	26	27
4	2	3	1	14	15	13
3	3	1	2	12	10	22
4	2	3	1	14	15	13
0	1	2	3	1	2	3
11	3	1	2	36	34	35

Λ	κ_i or κ'_i			Θ_i		
1	1	2	3	4	5	6
9	2	3	1	29	30	28
5	3	1	2	18	16	17
9	1	2	3	28	29	30
4	2	3	1	14	15	13
2	3	1	2	9	7	8
4	2	3	1	14	15	13
1	1	2	3	4	5	6
10	3	1	2	33	31	32

Table 4.22: Tables for Method 4 performed on f_4 with factor $(2r+1) = 3$ (cycles)

From the tables we get an EMT for a new graph that shown in Figure 4.37.

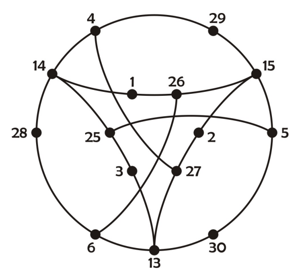


Figure 4.37: EMT labeling for new graph from f_4

Due to limited space in the graph, the edge labels are not included in the figure. They can be found in the tables if required.

Example: From f_4 using factor $(2r+1) = 5$

The table for the path is

Λ	κ_i					Θ_i				
2	1	2	3	4	5	41	42	43	44	45
4	3	4	5	1	2	8	9	10	6	7
13	5	3	1	4	2	35	33	31	34	32

Table 4.23: Table for Method 4 performed on f_4 with factor $(2r+1) = 5$ (paths)

The tables for the cycles are

Λ	κ_i or κ_i'					Θ_i					Λ	κ_i or κ_i'					Θ_i				
2	1	2	3	4	5	1	2	3	4	5	2	1	2	3	4	5	6	7	8	9	10
4	3	4	5	1	2	43	44	45	41	42	4	3	4	5	1	2	48	49	50	46	47
13	5	3	1	4	2	40	38	36	39	37	13	5	3	1	4	2	30	28	26	29	27
2	1	2	3	4	5	41	42	43	44	45	2	1	2	3	4	5	46	47	48	49	50
4	3	4	5	1	2	23	24	25	21	22	4	3	4	5	1	2	23	24	25	21	22
13	5	3	1	4	2	20	18	16	19	17	13	5	3	1	4	2	15	13	11	14	12
2	3	4	5	1	2	23	24	25	21	22	2	3	4	5	1	2	23	24	25	21	22
4	1	2	3	4	5	1	2	3	4	5	4	1	2	3	4	5	6	7	8	9	10
13	5	3	1	4	2	60	58	56	59	57	13	5	3	1	4	2	55	53	51	54	52

Table 4.24: Tables for Method 4 performed on f_4 with factor $(2r+1) = 5$ (cycles)

From the tables we get an EMT for a new graph that is shown in Figure 4.38.

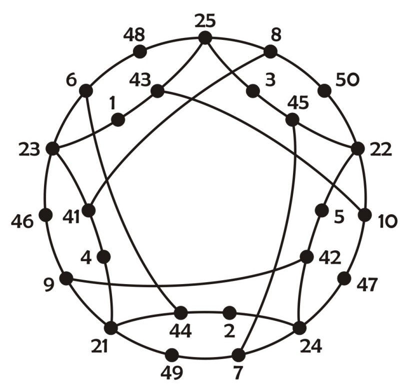


Figure 4.38: EMT labeling for new graph from f_4

Again, due to limited space in the graph, the edge labels are not included in the figure. They can be found in the tables if required.

Example: From f_6 using factor $(2r+1)=3$

In [22], EMT labeling of f_6 with $k=26$ given as shown in Figure 4.39.

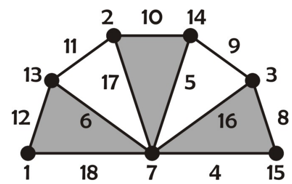


Figure 4.39: EMT labeling for f_6

In performing Method 4 to this graph, treat the shaded area as cycles, so we have 3 cycles and 2 paths. The tables are

Λ	κ_i or κ'_i			Θ_i		
0	1	2	3	1	2	3
12	2	3	1	38	39	37
11	3	1	2	36	34	35
12	1	2	3	37	38	39
6	2	3	1	20	21	19
5	3	1	2	18	16	17
6	2	3	1	20	21	19
0	1	2	3	1	2	3
17	3	1	2	54	52	53

Λ	κ_i or κ'_i			Θ_i		
1	1	2	3	4	5	6
13	2	3	1	41	42	40
9	3	1	2	30	28	29
13	1	2	3	40	41	42
6	2	3	1	20	21	19
4	3	1	2	15	13	14
6	2	3	1	20	21	19
1	1	2	3	4	5	6
16	3	1	2	51	49	50

Λ	κ_i or κ'_i			Θ_i		
2	1	2	3	7	8	9
14	2	3	1	44	45	43
7	3	1	2	24	22	23
14	1	2	3	43	44	45
6	2	3	1	20	21	19
3	3	1	2	12	10	11
6	2	3	1	20	21	19
2	1	2	3	7	8	9
15	3	1	2	48	46	47

Table 4.25: Tables for Method 4 performed on f_6 with factor $(2r+1)=5$ (cycles)

Λ	κ_i			Θ_i		
12	1	2	3	37	38	39
1	2	3	1	5	6	4
10	3	1	2	33	31	32

Λ	κ_i			Θ_i		
13	1	2	3	40	41	42
2	2	3	1	8	9	7
8	3	1	2	27	25	26

Table 4.26: Tables for Method 4 performed on f_6 with factor $(2r+1)=5$ (paths)

From the tables we get an EMT for a new graph that shown in Figure 4.40.

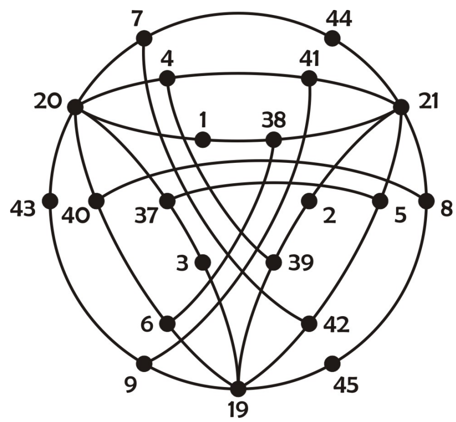


Figure 4.40: EMT labeling for new graph from f_6

For convenience let us call the graphs resulting from applying Method 4 to fan graphs *dreamcatchers* $D(l, m, c)$, where l, m and c denote the number of layers of cycles, common vertices shared by the cycles, and trellises of the dreamcatcher, respectively.

The cycles originated from the shaded triangles in the fan f_n, so we have $l = \frac{n}{2}$. The number of common vertices shared by the cycles is equal to the factor we use to apply the method with, that is, $2r + 1$, with the distance of each pair of common vertices is equal to 2.

The trellises originated from the leftover paths of the fan that is not a part of the shaded triangles. Hence for a dreamcatcher obtained from a fan f_n we have $c = (n-1)(2r+1)$. These trellises connect pairs of vertices from different layers of the cycles in such a manner that the resulting dreamcatcher has a rotational symmetry of order $(2r + 1)$.

Using this notation, Figures 4.37, 4.38 and 4.40 are dreamcatchers $D(2, 3, 3)$, $D(2, 5, 5)$ and $D(3, 3, 6)$, respectively.

We can summarize our results for dreamcatchers originated from fan graphs in the following theorem.

Theorem 4.10.1. *For $n \geq 2$ and $r \geq 0$, the dreamcatcher $D\left(\frac{n}{2}, (2r+1), (n-1)(2r+1)\right)$ has an EMT labeling.*

Proof. The result follows from applying Method 4 to the result in Theorem 2.12.2 for fan f_n. ■

Observe the similarity between fan, wheel and umbrella graph. From applying Method 4 to wheel and umbrella graphs, we also get dreamcatchers as the result, as described in the next two sections.

4.11 Method 4 Performed on Wheel W_n

Example: From W_6 using factor $(2r+1) = 3$

In [22], an EMT labeling of W_6 with $k = 26$ given as shown in Figure 4.41.

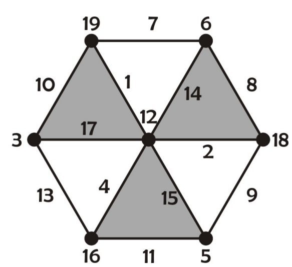


Figure 4.41: EMT labeling for W_6

In performing Method 4 to this graph, treat the shaded area as cycles, so we have 3 cycles and 3 paths. The tables are

Λ	κ_i			Θ_i			Λ	κ_i			Θ_i			Λ	κ_i			Θ_i		
18	1	2	3	55	56	57	17	1	2	3	52	53	54	15	1	2	3	46	47	48
5	2	3	1	17	18	16	4	2	3	1	14	15	13	2	2	3	1	8	9	7
6	3	1	2	21	19	20	8	3	1	2	27	25	26	12	3	1	2	39	37	38

Table 4.27: Tables for Method 4 performed on W_6 with factor $(2r+1) = 3$ (paths)

Λ	κ_i or κ'_i			Θ_i			Λ	κ_i or κ'_i			Θ_i			Λ	κ_i or κ'_i			Θ_i		
2	1	2	3	7	8	9	5	1	2	3	16	17	18	4	1	2	3	13	14	15
18	2	3	1	56	57	55	17	2	3	1	53	54	52	15	2	3	1	47	48	46
9	3	1	2	30	28	29	7	3	1	2	24	22	23	10	3	1	2	33	31	32
18	1	2	3	55	56	57	17	1	2	3	52	53	54	15	1	2	3	46	47	48
11	2	3	1	35	36	34	11	2	3	1	35	36	34	11	2	3	1	35	36	34
0	3	1	2	3	1	2	1	3	1	2	6	4	5	3	3	1	2	12	10	11
11	2	3	1	35	36	34	11	2	3	1	35	36	34	11	2	3	1	35	36	34
2	1	2	3	7	8	9	5	1	2	3	16	17	18	4	1	2	3	13	14	15
16	3	1	2	51	49	50	13	3	1	2	42	40	41	14	3	1	2	45	43	44

Table 4.28: Tables for Method 4 performed on W_6 with factor $(2r + 1) = 3$ (cycles)

From the tables we get an EMT for a new graph that shown in Figure 4.42.

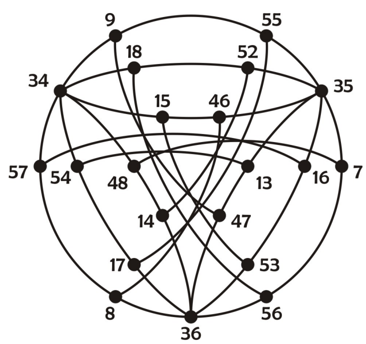


Figure 4.42: EMT labeling for new graph from W_6

Observe that Figure 4.42 really similar with dreamcatcher that we obtain in Section 4.10, the only difference is it have more trellises. Figure 4.42 is the dreamcatcher $D(3, 3, 9)$.

Unlike fans, wheels have the same number of shaded triangles and extra path that is not part of the shaded triangles. Hence the number of trellises is $c = n(2r + 1)$ instead of $(n - 1)(2r + 1)$.

We can summarize our results for dreamcatchers originated from wheel graphs in the following theorem.

Theorem 4.11.1. *For $n \equiv 6 \pmod 8$ and nonnegative integer r, the dreamcatcher $D\left(\frac{n}{2}, (2r+1), n(2r+1)\right)$ has an EMT labeling.*

Proof. The result follows from applying Method 4 to the result in Theorem 2.12.3 for wheel W_n. ■

4.12 Method 4 Performed on Umbrella $U(m, n)$

Now we apply Method 4 to the result about umbrella graph $U(m, n)$.

Example: From $U(4, 3)$ using factor $(2r + 1) = 3$

In [21], an SEMT labeling of $U(4, 3)$ with $k = 20$ given as shown in Figure 4.43.

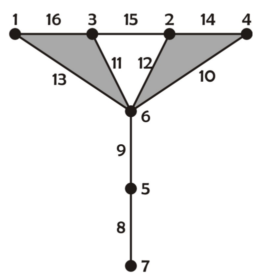


Figure 4.43: SEMT labeling for $U(4, 3)$

Again, treat the shaded area as cycles, so we have 2 cycles, a P_2 between the cycles and a path P_3 from the umbrella stick. The tables the are

Λ	κ_i			Θ_i		
5	1	2	3	16	17	18
4	2	3	1	14	15	13
8	3	1	2	27	25	26
4	1	2	3	13	14	15
6	2	3	1	20	21	19
7	3	1	2	24	23	22

Λ	κ_i			Θ_i		
2	1	2	3	7	8	9
1	2	3	1	5	6	4
14	3	1	2	45	43	44

Table 4.29: Tables for Method 4 performed on $U(4, 3)$ with factor $(2r + 1) = 3$ (paths)

Λ	κ_i or κ_i'			Θ_i		
0	1	2	3	1	2	3
2	2	3	1	8	9	7
15	3	1	2	48	46	47
2	1	2	3	7	8	9
5	2	3	1	17	18	16
10	3	1	2	33	31	32
5	2	3	1	17	18	16
0	1	2	3	1	2	3
12	3	1	2	39	37	38

Λ	κ_i or κ_i'			Θ_i		
1	1	2	3	4	5	6
3	2	3	1	11	12	10
13	3	1	2	42	40	41
3	1	2	3	10	11	12
5	2	3	1	17	18	16
9	3	1	2	30	28	29
5	2	3	1	17	18	16
1	1	2	3	4	5	6
11	3	1	2	36	34	35

Table 4.30: Tables for Method 4 performed on $U(4,3)$ with factor $(2r+1) = 3$ (cycles)

From the tables we get an SEMT for a new graph that shown in Figure 4.44.

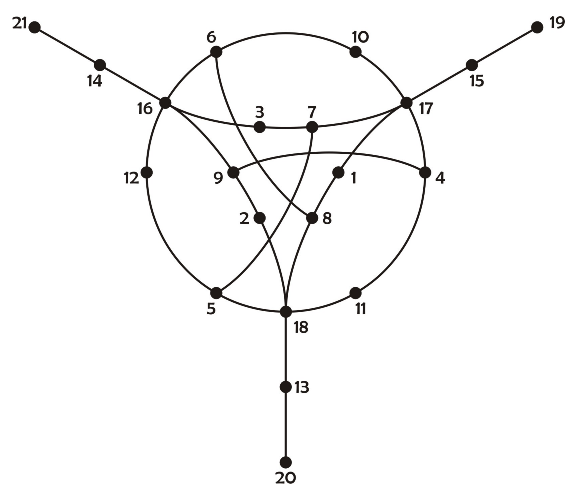


Figure 4.44: SEMT labeling for new graph from $U(4,3)$

The next example perform Method 4 for umbrella with greater value of m and n.

Example: From $U(6,4)$ using factor $(2r+1)=3$

In [21], an SEMT labeling of $U(6,4)$ with $k=29$ given as shown in Figure 4.45.

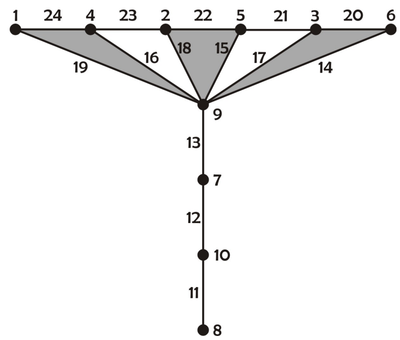


Figure 4.45: SEMT labeling for $U(6,4)$

Again, treat the shaded area as cycles, so we have 3 cycles, 2 paths P_2 between the cycles and a path P_4 from the umbrella stick. The tables are

Λ	κ_i or κ_i'			Θ_i		
0	1	2	3	1	2	3
3	2	3	1	11	12	10
23	3	1	2	72	70	71
3	1	2	3	10	11	12
8	2	3	1	26	27	25
15	3	1	2	48	46	47
8	2	3	1	26	27	25
0	1	2	3	1	2	3
18	3	1	2	57	55	56

Λ	κ_i or κ_i'			Θ_i		
1	1	2	3	4	5	6
4	2	3	1	14	15	13
21	3	1	2	66	64	65
4	1	2	3	13	14	15
8	2	3	1	26	27	25
14	3	1	2	45	43	44
8	2	3	1	26	27	25
1	1	2	3	4	5	6
17	3	1	2	54	52	53

Λ	κ_i or κ_i'			Θ_i		
2	1	2	3	7	8	9
5	2	3	1	17	18	16
19	3	1	2	60	58	59
5	1	2	3	16	17	18
8	2	3	1	26	27	25
13	3	1	2	42	40	41
8	2	3	1	26	27	25
2	1	2	3	7	8	9
16	3	1	2	51	49	50

Table 4.31: Tables for Method 4 performed on $U(6,4)$ with factor $(2r+1)=3$ (cycles)

Λ	κ_i			Θ_i		
3	1	2	3	10	11	12
1	2	3	1	5	6	4
22	3	1	2	69	67	68

Λ	κ_i			Θ_i		
4	1	2	3	13	14	15
2	2	3	1	8	9	7
20	3	1	2	63	61	62

Table 4.32: Tables for Method 4 performed on $U(6,4)$ with factor $(2r+1)=3$ (paths)

Λ	κ_i			Θ_i		
8	1	2	3	25	26	27
6	2	3	1	20	21	19
12	3	1	2	39	37	38
6	1	2	3	19	20	21
9	2	3	1	29	30	28
11	3	1	2	36	34	35
9	1	2	3	28	29	30
7	2	3	1	23	24	22
10	3	1	2	33	31	32

Table 4.33: Table for Method 4 performed on $U(6,4)$ with factor $(2r+1) = 3$ (stick)

From the tables we get an SEMT for a new graph shown in Figure 4.46.

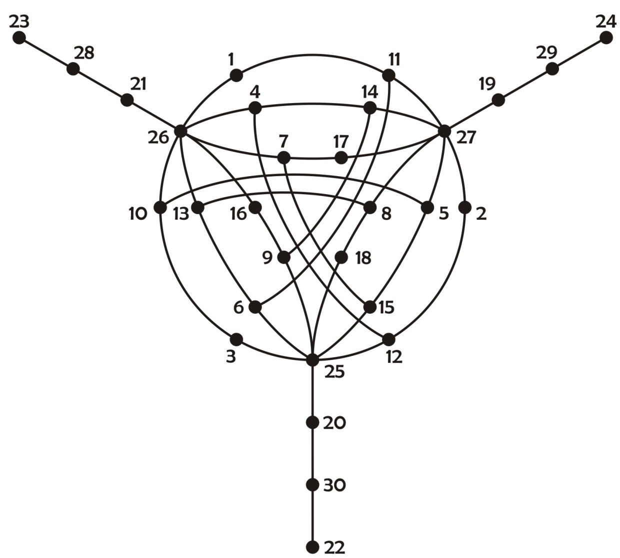


Figure 4.46: SEMT labeling for new graph from $U(6,4)$

How Method 4 works on an umbrella is similar with how it works for fan. The difference between a fan and an umbrella is the umbrella stick, which give extra feathers to the resulting dreamcatcher.

We denote *dreamcatcher with feathers* $D(l, m, c, t)$, which is a dreamcatcher $D(l, m, c)$ with m paths of length t rooting on the common vertices of the dreamcatcher as its feathers. Figure 4.44 and 4.46 are dreamcatchers with feather $D(2, 3, 3, 3)$ and $D(3, 3, 6, 4)$, respectively.

We can summarize our results for dreamcatchers originated from wheel graphs in the following theorem.

Theorem 4.12.1. *For nonnegative integer r, $m \geq 2$ and $n \geq 0$, the dreamcatcher $D\left(\frac{m}{2}, (2r+1), (m-1)(2r+1), n\right)$ has an SEMT labeling.*

Proof. The result follows from applying Method 4 to the result in Theorem 2.14.3 for umbrella $U(m, n)$. ■

4.13 $P_{2n}(+)N_m \to C_{(2n+1)(2r+1)}[+]N_m$

In performing the method to the graph $P_{2n}(+)N_m$, decompose the graph into a cycle with vertices $\{v_1, v_2, \ldots, v_{2n}, y_1\}$ and the paths $\{v_1, y_i, v_{2n}\}, i = 2, 3, \ldots, m$.

Example: $P_2(+)N_2 \to C_{15}[+]5N_1$

In [21], an SEMT labeling of $P_2(+)N_2$ was given as shown in Figure 4.47.

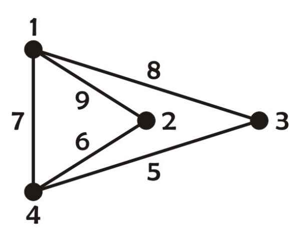


Figure 4.47: SEMT labeling for $P_2(+)N_2$

The tables are

Λ	κ_i or κ_i'					Θ_i				
0	1	2	3	4	5	1	2	3	4	5
1	3	4	5	1	2	8	9	10	6	7
8	5	3	1	4	2	45	43	41	44	42
1	1	2	3	4	5	6	7	8	9	10
3	3	4	5	1	2	18	19	20	16	17
5	5	3	1	4	2	30	28	26	29	27
3	3	4	5	1	2	18	19	20	16	17
0	1	2	3	4	5	1	2	3	4	5
6	5	3	1	4	2	35	33	31	34	32

Table 4.34: Table for $P_2(+)N_2 \to C_{15}[+]5N_1$ (cycle)

Λ	κ_i					Θ_i				
0	1	2	3	4	5	1	2	3	4	5
2	3	4	5	1	2	13	14	15	11	12
7	5	3	1	4	2	40	38	36	39	37
2	1	2	3	4	5	11	12	13	14	15
3	3	4	5	1	2	18	19	20	16	17
4	5	3	1	4	2	25	23	21	24	22

Table 4.35: Table for $P_2(+)N_2 \to C_{15}[+]5N_1$ (path)

From the tables we get an SEMT for $C_{15}[+]5N_1$ with $k = 54$ as shown below.

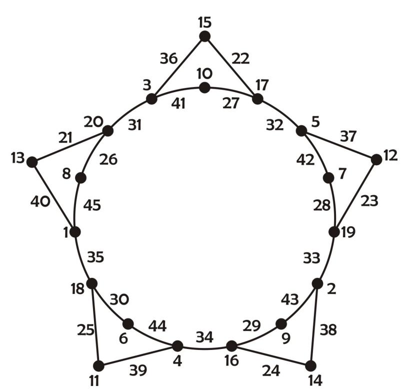


Figure 4.48: SEMT labeling for $C_{15}[+]5N_1$

Example: $P_2(+)N_3 \to C_9[+]3N_2$

In [21], an SEMT labeling of $P_2(+)N_3$ was given as shown in Figure 4.49.

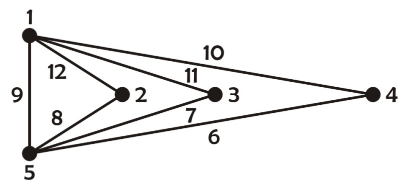


Figure 4.49: SEMT labeling for $P_2(+)N_3$

The tables are

Λ	κ_i or κ_i'			Θ_i		
0	1	2	3	1	2	3
1	2	3	1	5	6	4
11	3	1	2	36	34	35
1	1	2	3	4	5	6
4	2	3	1	14	15	13
7	3	1	2	24	22	23
4	2	3	1	14	15	13
0	1	2	3	1	2	3
8	3	1	2	27	25	26

Table 4.36: Table for $P_2(+)N_3 \to C_9[+]3N_2$ (cycle)

Λ	κ_i			Θ_i		
0	1	2	3	1	2	3
2	2	3	1	8	9	7
10	3	1	2	33	31	32
2	1	2	3	7	8	9
4	2	3	1	14	15	13
6	3	1	2	21	19	20

Λ	κ_i			Θ_i		
0	1	2	3	1	2	3
3	2	3	1	11	12	10
9	3	1	2	30	28	29
3	1	2	3	10	11	12
4	2	3	1	14	15	13
5	3	1	2	18	16	17

Table 4.37: Tables for $P_2(+)N_3 \to C_9[+]3N_2$ (paths)

From the tables we get an SEMT for $C_9[+]3N_2$ as shown in Figure 4.50.

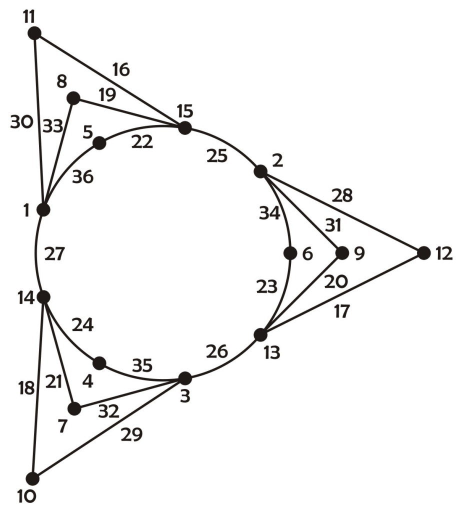


Figure 4.50: SEMT labeling for $C_9[+]3N_2$

Example: $P_4(+)N_2 \rightarrow C_{15}[+]3N_1$

In [21], an SEMT labeling of $P_4(+)N_2$ with $k = 15$ was given as shown in Figure 4.51.

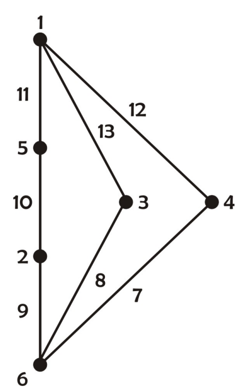


Figure 4.51: SEMT labeling for $P_4(+)N_2$

The tables for the cycle and the path are

Λ	κ_i or κ_i'			Θ_i		
0	1	2	3	1	2	3
2	2	3	1	8	9	7
12	3	1	2	39	37	38
2	1	2	3	7	8	9
5	2	3	1	17	18	16
7	3	1	2	24	22	23
5	1	2	3	16	17	18
1	2	3	1	5	6	4
8	3	1	2	27	25	26
1	2	3	1	5	6	4
4	1	2	3	13	14	15
9	3	1	2	30	28	29
4	2	3	1	14	15	13
0	1	2	3	1	2	3
10	3	1	2	33	31	32

Λ	κ_i			Θ_i		
0	1	2	3	1	2	3
3	2	3	1	11	12	10
11	3	1	2	36	34	35
3	1	2	3	10	11	12
5	2	3	1	17	18	16
6	3	1	2	21	19	20

Table 4.38: Tables for $P_4(+)N_2 \to C_{15}[+]3N_1$

From the tables we get an SEMT for $C_{15}[+]3N_1$ as shown in Figure 4.52.

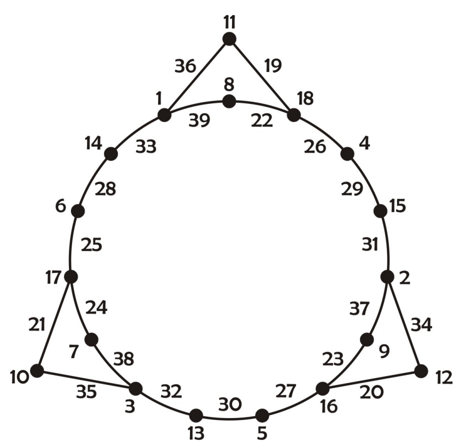


Figure 4.52: SEMT labeling for $C_{15}[+]3N_1$

Theorem 4.13.1. *The graph $C_{(2n+1)(2r+1)}[+]N_m$ has an SEMT labeling for all integers $m, n, r \geq 1$.*

Proof. The result follows from applying Method 4 to the graph in Theorem 2.13.1. ∎

4.14 Method 4 Performed on $(P_2 \cup mK_1) + N_2$

In [21], the planar graph $(P_2 \cup mK_1) + N_2$ has the vertex set $V = V(P_2) \cup V(mK_1) \cup V(N_2)$ where $V(P_2) = \{z_1, z_2\}, V(mK_1) = \{x_1, x_2, \ldots, x_m\}$ and $V(N_2) = \{y_1, y_2\}$. An example is given in Figure 4.53.

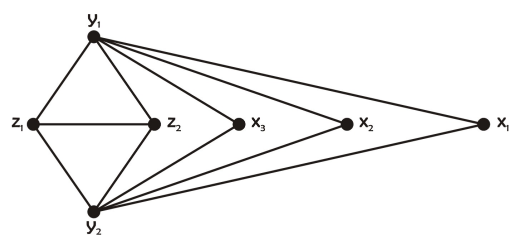


Figure 4.53: $(P_2 \cup 3K_1) + N_2$

Example: $(P_2 \cup 3K_1) + N_2 \rightarrow C_{12}[+]N_3$ **with 3 chords**

An SEMT labeling for this graph is given in Figure 4.54.

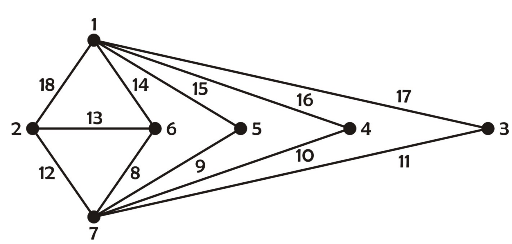


Figure 4.54: SEMT labeling for $(P_2 \cup 3K_1) + N_2$

In performing Method 4, we decompose the graph into C_4, $3P_3$, and P_2.

The tables are

Λ	κ_i			Θ_i		
0	1	2	3	1	2	3
3	2	3	1	11	12	10
11	3	1	2	36	34	35

Table 4.39: Tables for $(P_2 \cup 3K_1) + N_2 \to C_{12}[+]N_3$ with 3 chords (for P_2)

Λ	κ_i or κ_i'			Θ_i		
0	1	2	3	1	2	3
2	2	3	1	8	9	7
12	3	1	2	39	37	38
2	1	2	3	7	8	9
5	2	3	1	17	18	16
7	3	1	2	24	22	23
5	1	2	3	16	17	18
1	2	3	1	5	6	4
8	3	1	2	27	25	26
1	2	3	1	5	6	4
4	1	2	3	13	14	15
9	3	1	2	30	28	29
4	2	3	1	14	15	13
0	1	2	3	1	2	3
10	3	1	2	33	31	32

Table 4.40: Tables for $(P_2 \cup 3K_1) + N_2 \to C_{12}[+]N_3$ with 3 chords (for C_4)

Λ	κ_i			Θ_i		
0	1	2	3	1	2	3
1	2	3	1	5	6	4
11	3	1	2	36	34	35
1	1	2	3	4	5	6
4	2	3	1	14	15	13
7	3	1	2	24	22	23

Λ	κ_i			Θ_i		
0	1	2	3	1	2	3
2	2	3	1	8	9	7
10	3	1	2	33	31	32
2	1	2	3	7	8	9
4	2	3	1	14	15	13
6	3	1	2	21	19	20

Λ	κ_i			Θ_i		
0	1	2	3	1	2	3
3	2	3	1	11	12	10
9	3	1	2	30	28	29
3	1	2	3	10	11	12
4	2	3	1	14	15	13
5	3	1	2	18	16	17

Table 4.41: Tables for $(P_2 \cup 3K_1) + N_2 \to C_{12}[+]N_3$ with 3 chords (for $3P_3$)

From the tables we get an SEMT for $C_{12}[+]N_3$ with 3 chords as shown in Figure 4.55.

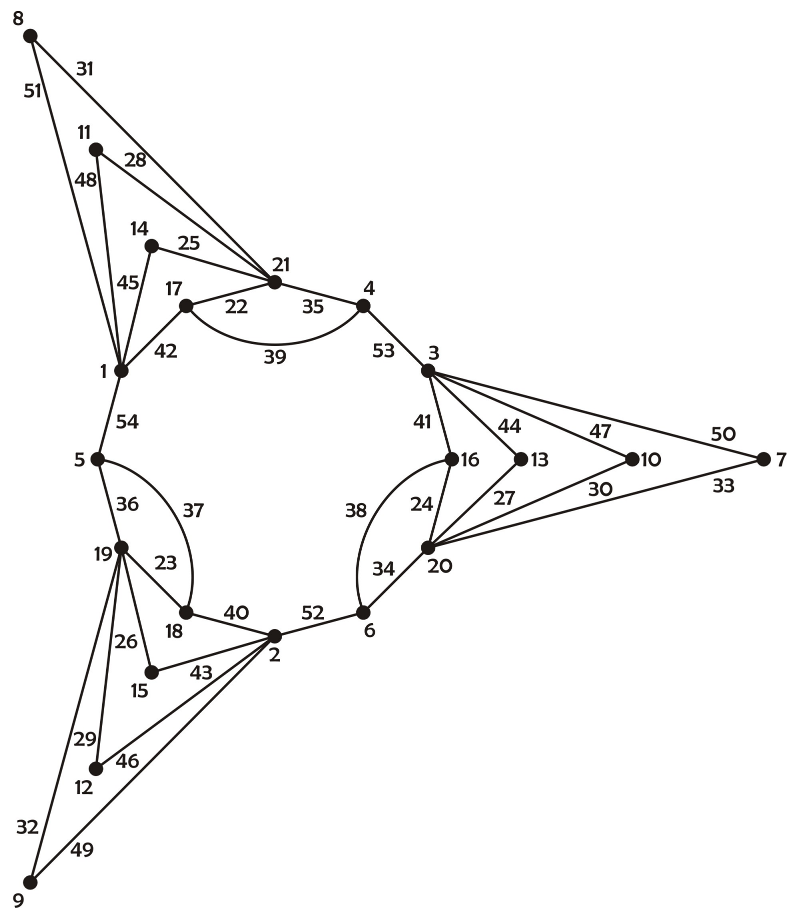


Figure 4.55: SEMT labeling for $C_{12}[+]N_3$ with 3 chords

Theorem 4.14.1. *The graph* $C_{(2n+1)(2r+1)}[+]N_m$ *with* $(2r+1)$ *chords has an SEMT labeling for all integers* $m, n, r \geq 1$.

Proof. The result follows from applying Method 4 to the result in Theorem 2.13.2. ■

4.15 Unions of Braids $mB(n)$

In this section we apply Method 4 to the result about braid graph $B(n)$ mentioned in Section 4.15.

Example: $B(3) \to 3B(3)$

In [21], an SEMT labeling of $B(3)$ with $k = 17$ was given as shown in Figure 4.56.

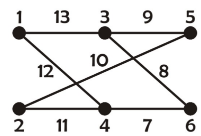


Figure 4.56: SEMT labeling for $B(3)$

Treat braid $B(3)$ as set of paths, consisting one central path with vertices label $(1, 3, 5, 2, 4, 6)$ and $2P_2$ with vertices label $(1, 4)$ and $(3, 6)$. For the second column of our method table, use all κ (not using κ' at all). Hence the tables for the method are

Λ	κ_i			Θ_i		
0	1	2	3	1	2	3
2	2	3	1	8	9	7
12	3	1	2	39	37	38
2	1	2	3	7	8	9
4	2	3	1	14	15	13
8	3	1	2	27	25	26
4	1	2	3	13	14	15
1	2	3	1	5	6	4
9	3	1	2	30	28	29
1	1	2	3	4	5	6
3	2	3	1	11	12	10
10	3	1	2	33	31	32
3	1	2	3	10	11	12
5	2	3	1	17	18	16
6	3	1	2	21	19	20

Λ	κ_i			Θ_i		
0	1	2	3	1	2	3
3	2	3	1	11	12	10
11	3	1	2	36	34	35

Λ	κ_i			Θ_i		
2	1	2	3	7	8	9
5	2	3	1	17	18	16
7	3	1	2	24	22	23

Table 4.42: Tables for $B(3) \to 3B(3)$

From the tables we get an SEMT for $3B(3)$ with $k = 48$ as shown in Figure 4.57.

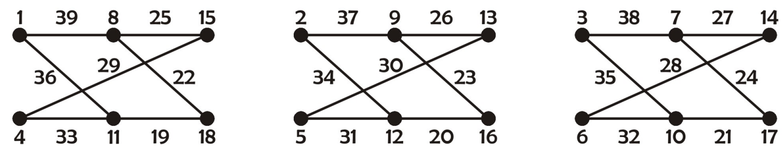


Figure 4.57: SEMT labeling for $3B(3)$

Applying Method 4 to braid graphs in general, we have the following theorem for unions of braids.

Theorem 4.15.1. *The union of braids $mB(n)$ has an SEMT labeling when m is odd.*

Proof. The result follows from applying Method 4 to the graph in Theorem 2.14.1. ■

4.16 Unions of Triangular Belts $mTB(\alpha)$

Next we apply Method 4 to the result about triangular belt $TB(\alpha)$.

Example: $TB\left(\downarrow^3\right) \to 3TB\left(\downarrow^3\right)$

In [21], an SEMT labeling of $TB\left(\downarrow^3\right)$ was given as shown in Figure 4.58.

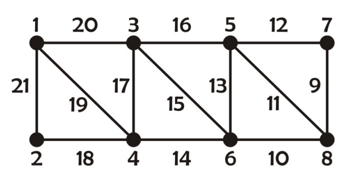


Figure 4.58: SEMT labeling for $TB\left(\downarrow^3\right)$

In applying Method 4 to triangular belt $TB\left(\downarrow^3\right)$, we treat the graph as a collection of paths, without considering cycles that are contained in it. Thus for the second column of our method table we can also just use κ. The tables for the method are

Λ	κ_i			Θ_i		
0	1	2	3	1	2	3
2	2	3	1	8	9	7
12	3	1	2	60	58	59
2	1	2	3	7	8	9
4	2	3	1	14	15	13
15	3	1	2	48	46	47
4	1	2	3	13	14	15
6	2	3	1	20	21	19
11	3	1	2	36	34	35

Λ	κ_i			Θ_i		
1	1	2	3	4	5	6
3	2	3	1	11	12	10
17	3	1	2	54	52	53
3	1	2	3	10	11	12
5	2	3	1	17	18	16
13	3	1	2	42	40	41
5	1	2	3	16	17	18
7	2	3	1	23	24	22
9	3	1	2	30	28	29

Table 4.43: Tables for $TB\left(\downarrow^3\right) \to 3TB\left(\downarrow^3\right)$ (horizontal paths)

Λ	κ_i			Θ_i		
0	1	2	3	1	2	3
1	2	3	1	5	6	4
20	3	1	2	63	61	62

Λ	κ_i			Θ_i		
2	1	2	3	7	8	9
3	2	3	1	11	12	10
16	3	1	2	50	48	49

Λ	κ_i			Θ_i		
4	1	2	3	13	14	15
5	2	3	1	17	18	16
12	3	1	2	39	37	38

Λ	κ_i			Θ_i		
6	1	2	3	19	20	21
7	2	3	1	23	24	22
8	3	1	2	27	25	26

Table 4.44: Tables for $TB\left(\downarrow^3\right) \to 3TB\left(\downarrow^3\right)$ (vertical paths)

For the diagonal paths, define new matrix κ'' as the matrix obtained by switching the second and third rows from κ.

$$\kappa'' = \begin{bmatrix} 1 & 2 & \dots & r+1 & r+2 & \dots & 2r & 2r+1 \\ 2r+1 & 2r & \dots & 1 & 2r-1 & \dots & 4 & 2 \\ r+1 & r+2 & \dots & 2r+1 & 1 & \dots & r-1 & r \end{bmatrix}$$

Hence the table for the diagonal paths are

Λ	κ''			Θ_i			Λ	κ''			Θ_i			Λ	κ''			Θ_i		
0	1	2	3	1	2	3	2	1	2	3	7	8	9	4	1	2	3	13	14	15
3	3	1	2	12	10	11	5	3	1	2	18	16	17	7	3	1	2	24	22	23
18	2	3	1	56	57	55	14	2	3	1	44	45	43	10	2	3	1	32	33	31

Table 4.45: Tables for $TB\left(\downarrow^3\right) \to 3TB\left(\downarrow^3\right)$ (diagonal paths)

From the tables we get an SEMT labeling for $3TB(3)$ with $k = 69$.

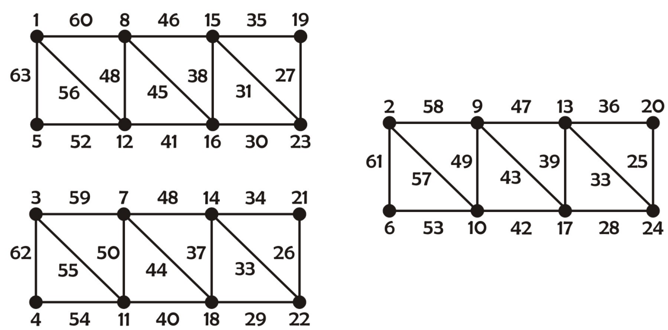


Figure 4.59: SEMT labeling for $3TB\left(\downarrow^3\right)$

Applying Method 4 to triangular belts in general, we have the following theorem for union of triangular belts.

Theorem 4.16.1. *For any $\alpha \in S^n, S = \{\uparrow, \downarrow\}, n > 1$ and odd m, the union of triangular belts $mTB(\alpha)$ has an SEMT labeling.*

Proof. The result follows from applying Method 4 to the graph in Theorem 2.14.2. ■

4.17 Jellyfish $J(m, n) \to$ Rambutan $R(t, m, n)$

Applying Method 4 to an SEMT labeling of a jellyfish $J(m, n)$ will give an SEMT labeling of the rambutan $R(t, m, n)$ for any odd t. Treat the body of the jellyfish as a cycle with one chord and all its tentacles as paths. The example will show how to obtain $R(3, 3, 2)$ from $J(2, 3)$.

Example: $J(2,3) \to R(3,3,2)$

In [21], an SEMT labeling of $J(2,3)$ with $k = 24$ was given as shown in Figure 4.60.

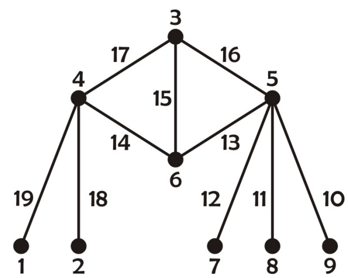


Figure 4.60: SEMT labeling for $J(2,3)$

Tables for the jellyfish are

Λ	κ_i or κ_i'			Θ_i		
3	1	2	3	10	11	12
2	2	3	1	8	9	7
16	3	1	2	51	49	50
2	1	2	3	7	8	9
4	2	3	1	14	15	13
15	3	1	2	48	46	47
4	1	2	3	13	14	15
5	2	3	1	17	18	16
12	3	1	2	39	37	38
5	2	3	1	17	18	16
3	1	2	3	10	11	12
13	3	1	2	42	40	41

Λ	κ_i			Θ_i		
2	1	2	3	7	8	9
5	2	3	1	17	18	16
14	3	1	2	45	43	44

Table 4.46: Tables for $J(2,3) \to R(3,3,2)$ (body)

Λ	κ_i			Θ_i		
3	1	2	3	10	11	12
0	2	3	1	2	3	1
18	3	1	2	57	55	56

Λ	κ_i			Θ_i		
3	1	2	3	10	11	12
1	2	3	1	5	6	4
17	3	1	2	54	52	53

Table 4.47: Tables for $J(2,3) \to R(3,3,2)$ (for $m = 2$ tentacles)

Λ	κ_i			Θ_i			Λ	κ_i			Θ_i			Λ	κ_i			Θ_i		
4	1	2	3	13	14	15	4	1	2	3	13	14	15	4	1	2	3	13	14	15
6	2	3	1	20	21	19	7	2	3	1	23	24	22	8	2	3	1	26	27	15
11	3	1	2	36	34	35	10	3	1	2	33	31	32	9	3	1	2	30	28	29

Table 4.48: Tables for $J(2,3) \to R(3,3,2)$ (for $n = 3$ tentacles)

From the tables we get an SEMT for $R(3,3,2)$ with $k = 69$ shown in Figure 4.61.

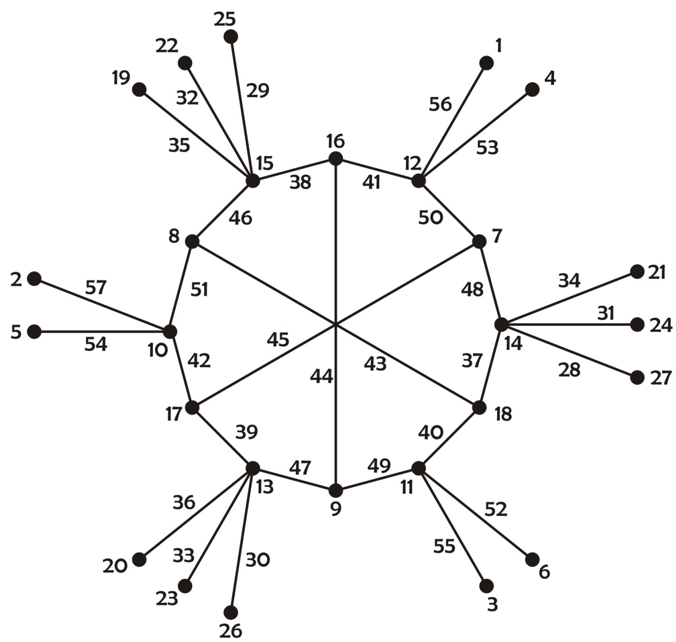


Figure 4.61: SEMT labeling for $R(3,3,2)$

We can summarize the result from this section in the theorem below.

Theorem 4.17.1. *For odd t and all $m,n \geq 0$ the graph $R(t,m,n)$ has an SEMT labeling.*

Proof. The result follows from applying Method 4 to the graph in Theorem 2.15.1. ∎

4.18 Incomplete Mongolian Ger $MT(m, h)$ to a Circus Tent

Applying Method 4 to the SEMT labeling of the incomplete Mongolian ger $MT(m, h)$ will give a different family of graph. Since the body of the ger is the Cartesian product $P_h \square P_m$, when we apply the method, it will extend the length of the ger to $P_h \square P_{m(2r+1)}$.

Meanwhile, since we only consider odd values of m the roof of the ger is a star $S_{(m+1)/2}$ which can be treated as a set of paths so that when we apply the method, it will multiply the number of identical roofs instead of extending it.

For convenience we call the resulting graph the *circus tent* $CT(m, h, t)$, where m and h are as defined in the incomplete Mongolian ger $MT(m, h)$ and t represents the number of roof. To see better how the method works we present the following example.

Example: $MT(3, 2) \to CT\,(9, 2, 3)$

In [21], an SEMT labeling of $MT(3, 2)$ with $k = 21$ was given as shown in Figure 4.62.

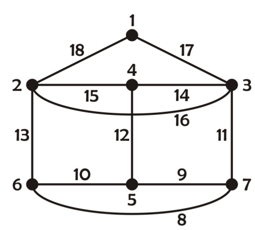


Figure 4.62: SEMT labeling for $MT(3, 2)$

The tables are

Λ	κ_i			Θ_i		
1	1	2	3	4	5	6
0	2	3	1	2	3	1
17	3	1	2	54	52	53

Λ	κ_i			Θ_i		
0	1	2	3	1	2	3
2	2	3	1	8	9	7
16	3	1	2	51	49	50

Table 4.49: Tables for $MT(3, 2) \to CT\,(9, 2, 3)$ (roof)

Λ	κ_i			Θ_i		
1	1	2	3	4	5	6
5	2	3	1	17	18	16
12	3	1	2	39	37	38

Λ	κ_i			Θ_i		
3	1	2	3	10	11	12
4	2	3	1	14	15	13
11	3	1	2	36	34	35

Λ	κ_i			Θ_i		
2	1	2	3	7	8	9
6	2	3	1	20	21	19
10	3	1	2	33	31	32

Table 4.50: Tables for $MT(3,2) \to CT\,(9,2,3)$ (vertical paths)

Λ	κ_i or κ'_i			Θ_i		
1	1	2	3	4	5	6
3	2	3	1	11	12	10
14	3	1	2	45	43	44
3	1	2	3	10	11	12
2	2	3	1	8	9	7
13	3	1	2	42	40	41
2	2	3	1	8	9	7
1	1	2	3	4	5	6
15	3	1	2	48	46	47

Λ	κ_i or κ'_i			Θ_i		
5	1	2	3	16	17	18
4	2	3	1	14	15	13
9	3	1	2	30	28	29
4	1	2	3	13	14	15
6	2	3	1	20	21	19
8	3	1	2	27	25	26
6	2	3	1	20	21	19
5	1	2	3	16	17	18
7	3	1	2	24	22	23

Table 4.51: Tables for $MT(3,2) \to CT\,(9,2,3)$ (cycles)

From the tables we get an SEMT for $CT\,(9,2,3)$ as shown in Figure 4.63.

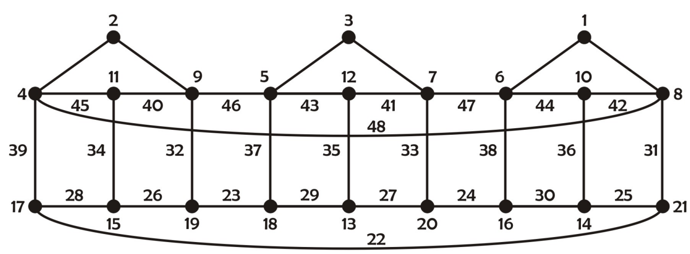


Figure 4.63: SEMT labeling for $CT\,(9,2,3)$

Thus we can summarize the result from this section into the theorem below.

Theorem 4.18.1. *For each $r \geq 0$, $h \geq 2$ and odd values of $m \geq 3$, the circus tent $CT\,((2r+1)m, h, 2r+1)$ has an SEMT labeling.*

Proof. The result follows from applying Method 4 to the graph in Theorem 2.15.2. ■

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Appendix A

Summary of Known VMT and SVMT Labeling

In this Appendix, we give tables of known results in VMT and SVMT labeling. All information is taken from the comprehensive survey by Gallian [6] combined with the dissertation written by Sugeng [24].

Table A.1: Summary of Vertex-magic Total Labelings

Graph	**Labeling**	**Notes**
C_n	VMT	
P_n	VMT	for $n > 2$
$K_{m,m}$	VMT	$m > 1$
$K_{m,m} - e$	VMT	$m > 2$
$K_{m,n}$	VMT	iff $\|m - n\| \leq 1$
K_n	VMT	for n odd for $n \equiv 2 \pmod 4, n > 2$

Table A.2: Summary of Vertex-magic Total Labelings (cont.)

Graph	**Labeling**	**Notes**
nK_3	VMT	iff $n \neq 2$
mK_n	VMT	$m \geq 1, n \geq 4$
Petersen $P(n,k)$	VMT	
prisms $C_n \square P_2$	VMT	
W_n	VMT	iff $n \leq 11$
fan f_n	VMT	iff $n \leq 10$
friendship graphs Fr_n	VMT	if the number of triangles ≤ 3
$G + H$	VMT	$\lvert V(G)\rvert = \lvert V(H)\rvert$ and $G \cup H$ is VMT
$St(a_1, a_2, \ldots, a_n)$	VMT	
tree with n internal vertices and more than $2n$ leaves	not VMT	
nG	VMT	n odd, G regular of even degree
nG	VMT	G regular of odd degree, but not K_1
$C_n \square C_{2m+1}$	VMT	
$K_5 \square C_{2n+1}$	VMT	
$G \square C_{2n}$	VMT	G is $2r + 1$ regular VMT
$G \square K_5$	VMT	G is $2r + 1$ regular VMT
$G \square H$	VMT	G is r-regular VMT, r odd or r even and $\lvert H\rvert$ odd, H is $2s$-regular supermagic

Table A.3: Summary of Super Vertex-magic Total Labelings

Graph	**Labeling**	**Notes**
P_n	SVMT	iff $n > 1$ is odd, $n \geq 3$
C_n	SVMT	iff n is odd
$K_{1,n}$	SVMT	iff $n = 1$
mC_n	SVMT	iff m and n are odd
W_n	not SVMT	
ladders L_n	not SVMT	
friendship graphs Fr_n	not SVMT	
$K_{m,n}$	not SVMT	
K_{4n}	SVMT	
dragons	SVMT	iff order is even
Knödel graphs	SVMT	$n \equiv 0 \pmod 4$
graphs with minimum degree 1	not SVMT	

Appendix B

Summary of Known EMT and SEMT Labeling

In this Appendix, we give tables of known results in EMT and SEMT labeling. All information is taken from the comprehensive survey by Gallian [6] combined with the dissertation written by Sugeng [24]. The question mark "?" denotes that the statement is a conjecture.

Table B.1: Summary of Edge-magic Total Labelings

Graph	**Labeling**	**Notes**
P_n	EMT	
C_n	EMT	for $n \geq 3$
K_n	EMT	iff $n = 1, 2, 3, 4, 5, 6$
$K_{m,n}$	EMT	
crowns $C_n \odot K_1$	EMT	

Table B.2: Summary of Edge-magic Total Labelings (cont.)

Graph	**Labeling**	**Notes**
wheels W_n	EMT	iff $n \equiv 3 \pmod 4$
pan graph ($\{n,1\}$-tadpole graph)	EMT	
fans f_n	EMT	
trees	EMT?	
(n,e)-graphs	not EMT	if e even and $n+e \equiv 2 \pmod 4$
nP_2	EMT	iff n odd
$P_n + K_1$	EMT	
$P_n \square C_3$	EMT	$n \geq 2$
r-regular graphs	not EMT	if r odd and $n \equiv 4 \pmod 8$
r-regular (n,e) graphs	not EMT	if $r = 2^t s + 1, t > 0$, e even and $2^t + 2$ divides n
$P_3 \cup nK_2$ and $P_5 \cup nK_2$	EMT	
$P_4 \cup nK_2$	EMT	n odd
nP_i	EMT	n odd, $i = 3, 4, 5$
nP_3	EMT	
$2P_n$	EMT	
$P_1 \cup P_2 \cup \ldots \cup P_n$	EMT	
$mK_{1,n}$	EMT	
$C_m \odot \overline{K_n}$	EMT	
unicyclic graphs	EMT?	
$K_1 \odot nK_2$	EMT	n even
$K_2 \square \overline{K_n}$	EMT	
nK_3	EMT	
binary trees	EMT	
generalized Petersen graph $P(n,k)$	EMT	

Table B.3: Summary of Edge-magic Total Labelings (cont.)

Graph	**Labeling**	**Notes**
books B_n	EMT	
odd cycles with pendant edges attached to one vertex	EMT	
$P_m \square C_n$	EMT	n odd, $n \geq 3$
$P_m \square P_2$	EMT	m odd, $m \geq 3$
$P_2 \square C_n$	not EMT	
$K_{1,m} \cup K_{1,n}$	EMT	iff mn is even
$G \odot \overline{K_n}$	EMT	G is EMT 2-regular graph

Table B.4: Summary of Super Edge-magic Total Labelings

Graph	**Labeling**	**Notes**
C_n	SEMT	iff n is odd
caterpillars	SEMT	
$K_{m,n}$	SEMT	iff $m = 1$ or $n = 1$
K_n	SEMT	iff $n = 1, 2, 3$
trees	SEMT?	
nK_2	SEMT	iff n is odd
nG	SEMT	if G is a bipartite or tripartite SEMT graph and n is odd
$K_{1,m} \cup K_{1,n}$	SEMT	if m is a multiple of $n + 1$
$K_{1,2} \cup K_{1,n}$	SEMT	iff n is a multiple of 3

Table B.5: Summary of Super Edge-magic Total Labelings (cont.)

Graph	**Labeling**	**Notes**
$K_{1,3} \cup K_{1,n}$	SEMT	iff n is a multiple of 4
$P_m \cup K_{1,n}$	SEMT	if m even, $m \geq 4$
$2P_n$	SEMT	iff $n \neq 2$ and $n \neq 3$
$2P_{4n}$	SEMT	
$K_{1,m} \cup 2nK_{1,2}$	SEMT	
$C_3 \cup C_n$	SEMT	iff n even, $n \geq 6$
$C_4 \cup C_n$	SEMT	iff n odd, $n \geq 5$
$C_5 \cup C_n$	SEMT	iff n even, $n \geq 4$
$C_m \cup C_n$	SEMT	if $m \geq 6$ and n odd, $n \geq \frac{m}{2} + 2$
$C_m \cup C_n$	SEMT?	iff $m + n \geq 9$ and $m + n$ odd
$C_4 \cup P_n$	SEMT	iff $n \neq 3$
$C_5 \cup P_n$	SEMT	if $n \neq 4$
$C_m \cup P_n$	SEMT	if m even, $m \geq 6$ and $n \geq \frac{m}{2} + 2$
$P_m \cup P_n$	SEMT	iff $(m, n) \neq (2, 2)$ or $(3, 3)$
corona $C_n \odot \overline{K_m}$	SEMT	if $n \geq 3$
$St(m, n)$	SEMT	$n \equiv 0 \pmod ()m + 1)$
$St(1, k, n)$	SEMT	$k \in \{1, 2, n\}$
$St(2, k, n)$	SEMT	$k \in \{2, 3\}$
$St(1, 1, k, n)$	SEMT	$k \in \{2, 3\}$
$St(k, 2, 2, n)$	SEMT	$k \in \{1, 2\}$
$St(a_1, a_2, \ldots, a_n)$	SEMT?	n odd, $n > 1$
${}^tC_{4m}$	SEMT	
${}^tC_{4m+1}$	SEMT	
friendship graph f_{2n}	SEMT	iff $n \in 3, 4, 5, 7$

Table B.6: Summary of Super Edge-magic Total Labelings (cont.)

Graph	**Labeling**	**Notes**
generalized Petersen Graph $P(n, 2)$	SEMT	if n odd, $n \geq 3$
nP_3	SEMT	if n even, $n \geq 4$
P_n^2	SEMT	
$K_2 \times C_{2n+1}$	SEMT	
$P_3 \cup kP_2$	SEMT	for all k
kP_n	SEMT	if k is odd
$k(P_2 \cup P_n)$	SEMT	if k is odd and $n \in \{3, 4\}$
fans F_n	SEMT	iff $n \leq 6$
books B_n	SEMT	if n is even
books B_n	SEMT?	if $n \equiv 5 \pmod 8$
trees with α labeling	SEMT	
$P_{2m+1} \square P_2$	SEMT	
$C_{2m+1} \square P_m$	SEMT	
$G \odot \overline{K_n}$	SEMT	G is SEMT 2-regular graph
$C_m \odot \overline{K_n}$	SEMT	
join of K_1 with any subgraph of a star	SEMT	
join of two nontrivial graphs one has two vertices and their union has exactly one edge	SEMT	
G is connected (n, e) graph	SEMT	G exists iff $p - 1 \leq q \leq 2p - 3$
G is connected 3-regular graph on p vertices	SEMT	iff $p \equiv 2 \pmod 4$
$nK_2 + nK_2$	not SEMT	

Appendix C

Summary of New Results

In this Appendix, we give tables of new results that we stated in this thesis. In the tables we use "S." to denote section and "Label" to denote the type of the magic labeling.

Table C.1: Summary of New Results

S.	**Label**	**Graph**	**Notes**
3.3	VMT	mC_n	$m \equiv 2 \pmod 4$ and either $n \equiv 2 \pmod 4$, $n \equiv 4 \pmod 8$ or $n \equiv 8 \pmod{16}$
	VMT	mC_n	$m \equiv 4 \pmod 8$ and $n \equiv 4 \pmod 8$
3.4	SVMT	$m\left(C_{3(2r+1)} \cup C_{n_2(2r+1)}\right)$	$n_2 \geq 6$, n_2 even
	SVMT	$m\left(C_{n_1(2r+1)} \cup C_{4(2r+1)}\right)$	$n_1 \geq 5$, n_1 odd
	SVMT	$m\left(C_{5(2r+1)} \cup C_{n_2(2r+1)}\right)$	$n_2 \geq 4$, n_2 even
	SVMT	$m\left(C_{n_1(2r+1)} \cup C_{n_2(2r+1)}\right)$	m odd, n_1 even, $n_2 \geq 4$, and either: n_2 odd and $n_2 \geq \frac{n_1}{2} + 2$, or $\gcd(n_2, 3) \neq 1$, or $\gcd(n_2, 5) \neq 1$.
	VMT	$m\left(C_{6(2r+1)} \cup C_{4(2r+1)}\right)$	$r \geq 0$, m odd

Table C.2: Summary of New Results (cont.)

S.	**Label**	**Graph**	**Notes**
3.4	SVMT	$m\left(C_{4(2r+1)} \cup (2t-1)C_{3(2r+1)}\right)$	$r \geq 0$, m odd and $t \geq 3$
	SVMT	$m\left(C_{5(2r+1)} \cup (2t)C_{3(2r+1)}\right)$	$r \geq 0$, m odd and $t \geq 2$
	SVMT	$m\left(C_{7(2r+1)} \cup (2t)C_{3(2r+1)}\right)$	$r \geq 0$, m odd and $t \geq 1$
4.2	SEMT	$m\left(C_{3(2r+1)} \cup (2r+1)P_{n_2}\right)$	$r \geq 0$, m odd and $n_2 \geq 6$
	SEMT	$m\left(C_{4(2r+1)} \cup (2r+1)P_{n_2}\right)$	$r \geq 0$, m odd and $n_2 \neq 3$
	SEMT	$m\left(C_{5(2r+1)} \cup (2r+1)P_{n_2}\right)$	$r \geq 0$, m odd and $n_2 \geq 4$
	SEMT	$m\left(C_{n_1(2r+1)} \cup (2r+1)P_{n_2}\right)$	$r \geq 0$, m odd, $n_1 \geq 4$, $n_2 \geq \frac{n_1}{2}+2$
4.3	SEMT	${}^{[(2r+1)^h]t_h}C_{n(2r+1)^h}$	for some $a \geq 0$ and $t_a \geq 0$, if ${}^tC_n$ has an SEMT
4.4	EMT	$m\left(C_{4n} \circ \overline{K}_2\right)$	both m and n odd
4.6	SEMT	$m\left(P_2 \,\square\, P_n\right)$	m and n both odd
4.7	SEMT	$(2r+1)\left(P_m \cup P_n\right)$	$r \geq 0$, $(m,n) \neq (2,2)$ or $(3,3)$
	SEMT	$(2r+1)\left(P_3 \cup mP_2\right)$	$r \geq 0$, $m \geq 0$
	SEMT	mP_n	$m \equiv 2 \pmod 4$, $n \neq 2,3$
	SEMT	mP_{4n}	$n > 1$, $m \equiv 2 \pmod 4$
4.8	EMT	(n,t)-tadpole	$n \in \{3,4\}, t > 0$
	EMT	$(n,2)$-tadpole	n odd
	SEMT	mutated $((2r+1),(2r+1)n,1)$-tadpole	$r \geq 0$, n odd
	EMT	mutated $((2r+1),(2r+1)n,1)$-tadpole	$r \geq 0$, n even
	SEMT	mutated $((2r+1),(2r+1)n,2)$-tadpole	$r \geq 0$, n even
	EMT	mutated $((2r+1),(2r+1)n,2)$-tadpole	$r \geq 0$, n odd
	EMT	mutated $((2r+1),(2r+1)n,t)$-tadpole	$r, t \geq 0$, $n \in \{3,4\}$

Table C.3: Summary of New Results (cont.)

S.	**Label**	**Graph**	**Notes**
4.9	SEMT	$C_{3m}(n,m)$	m odd, $n \in \{3,4,5,7\}$
4.10	EMT	$D\left(\frac{n}{2},(2r+1),(n-1)(2r+1)\right)$	$n \geq 2$, $r \geq 0$
4.11	EMT	$D\left(\frac{n}{2},(2r+1),n(2r+1)\right)$	$n \equiv 6 \pmod 8$, $r \geq 0$
4.12	SEMT	$D\left(\frac{m}{2},(2r+1),(m-1)(2r+1),n\right)$	$m \geq 2$, $n \geq 0$, $r \geq 0$
4.13	SEMT	$C_{(2r+1)(2n+1)}[+]N_m$	$m,n,r \geq 1$
4.14	SEMT	$C_{(2r+1)(2n+1)}[+]N_m$ with $(2r+1)$ pendant edges	$m,n,r \geq 1$
4.15	SEMT	$mB(n)$	m odd
4.16	SEMT	$mTB(\alpha)$	$\alpha \in S^n, S=\{\uparrow,\downarrow\}, n>1$ and m odd
4.17	SEMT	$R(t,m.n)$	$m,n \geq 0$, t odd
4.18	SEMT	$CT\left((2r+1)m,h,2r+1\right)$	odd $m \geq 3$, $r \geq 0$, $h \geq 2$